Semisimple algebras of almost minimal rank over the reals

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Abstract

A famous lower bound for the bilinear complexity of the multiplication in associative algebras is the Alder–Strassen bound. Algebras for which this bound is tight are called algebras of minimal rank. After 25 years of research, these algebras are now well understood. We here start the investigation of the algebras for which the Alder–Strassen bound is off by one. As a first result, we completely characterize the semisimple algebras over $\mathbb{R}$ whose bilinear complexity is by one larger than the Alder–Strassen bound. Furthermore, we characterize all algebras A (with radical) of minimal rank plus one over $\mathbb{R}$ for which $A/\operatorname{rad} A$ has minimal rank plus one. The other possibility is that $A/\operatorname{rad} A$ has minimal rank. For this case, we only present a partial result.

Key words: Bilinear complexity, associative algebras, algebras of minimal rank.

1. Introduction

A central problem in algebraic complexity theory is the question about the costs of multiplication in associative algebras. Let A be a finite dimensional associative k-algebra with unity 1. By fixing a basis of A, say $v_1, \dots, v_N$, we can define a set of bilinear forms corresponding to the multiplication in A. If $v_\mu v_\nu = \sum_{\kappa=1}^{N} \alpha_{\mu,\nu}^{(\kappa)} v_\kappa$ for $1 \le \mu, \nu \le N$ with *structural constants* $\alpha_{\mu,\nu}^{(\kappa)} \in k$, then these constants and the identity

$$\left(\sum_{\mu=1}^{N} X_\mu v_\mu\right)\left(\sum_{\nu=1}^{N} Y_\nu v_\nu\right) = \sum_{\kappa=1}^{N} b_\kappa(X,Y) v_\kappa$$

define the desired bilinear forms $b_1, \dots, b_N$. The *bilinear complexity* or *rank* of these bilinear forms $b_1, \dots, b_N$ is the smallest number of essential bilinear

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multiplications necessary and sufficient to compute $b_1, \dots, b_N$ from the indeterminates $X_1, \dots, X_N$ and $Y_1, \dots, Y_N$. More precisely, the bilinear complexity of $b_1, \dots, b_N$ is the smallest number r of products $p_\rho = u_\rho(X_i) \cdot v_\rho(Y_j)$ with linear forms u_ρ and v_ρ in the X_i and Y_j, respectively, such that $b_1, \dots, b_N$ are contained in the linear span of $p_1, \dots, p_r$. From this characterization, it follows that the bilinear complexity of $b_1, \dots, b_N$ does not depend on the choice of $v_1, \dots, v_N$, thus we may speak about the bilinear complexity of (the multiplication in) A. For a modern introduction to this topic and to algebraic complexity theory in general, we recommend [9].

A fundamental lower bound for the rank of an associative algebra A is the so-called Alder–Strassen bound [1]. It states that the rank of A is bounded from below by twice the dimension of A minus the number of twosided ideals in A. This bound is sharp in the sense that there are algebras for which equality holds. For instance, for $A = k^{2\times 2}$, we get a lower bound of 7, since $k^{2\times 2}$ is a simple algebra and has only one twosided ideal (other than $k^{2\times 2}$). 7 is a sharp bound, since we can multiply 2×2–matrices with 7 multiplications by Strassen's algorithm.

An algebra A has minimal rank if the Alder–Strassen bound is sharp, that is, the rank of A equals twice the dimension minus the number of two-sided ideals. After 25 years of effort [10, 11, 8, 12], the algebras of minimal rank were characterized in terms of their algebraic structure [6]: An algebra over some field k has minimal rank if and only if

$$A \cong C_1 \times \cdots \times C_s \times k^{2\times 2} \times \cdots \times k^{2\times 2} \times A'$$

where $C_1, \dots, C_s$ are local algebras of minimal rank with $\dim(C_\sigma / \operatorname{rad} C_\sigma) \geq 2$ (as determined in [8]) and $\#k \geq 2 \dim C_\sigma - 2$, and A' is an algebra of minimal rank such that $A' / \operatorname{rad} A' \cong k^t$ for some t. Such an algebra A' has minimal rank if and only if there exist $w_1, \dots, w_m \in \operatorname{rad} A$ with $w_i w_j = 0$ for $i \neq j$ such that

$$\operatorname{rad} A = \mathsf{L}_A + A w_1 A + \cdots + A w_m A = \mathsf{R}_A + A w_1 A + \cdots + A w_m A$$

and $\#k \geq 2N(A) - 2$. Here L_A and R_A denote the left and right annihilator of $\operatorname{rad} A$, respectively, and $N(A)$ is the largest natural number s such that $(\operatorname{rad} A)^s \neq \{0\}$.

1.1. Model of computation

In the remainder of this work, we use a coordinate-free definition of rank, which is more appropriate when dealing with algebras of minimal rank, see [9, Chap. 14]. For a vector space V, V^* denotes the dual space of V, that is, the vector space of all linear forms on V. For a set of vectors U, $\operatorname{lin} U$ denotes the linear span of U, i.e., the smallest vector space that contains U.

Definition 1. *Let k be a field, U, V, and W be finite dimensional vector spaces over k, and $\phi : U \times V \to W$ be a bilinear map.*

1. *A sequence* $\beta = (f_1, g_1, w_1; \ldots; f_r, g_r, w_r)$ *such that* $f_\rho \in U^*$, $g_\rho \in V^*$, *and* $w_\rho \in W$ *is called a* bilinear computation *of length* r *for* ϕ *if*

$$\phi(u, v) = \sum_{\rho=1}^{r} f_\rho(u) g_\rho(v) w_\rho \quad \text{for all } u \in U, v \in V.$$

2. *The length of a shortest bilinear computation for* ϕ *is called the* bilinear complexity *or the* rank *of* ϕ *and is denoted by* $R(\phi)$ *or* $R_k(\phi)$ *if we want to stress the underlying field* k.
3. *If* A *is a finite dimensional associative* k*-algebra with unity, then the rank of* A *is defined as the rank of the multiplication map of* A*, which is a bilinear map* $A \times A \to A$. *The rank of* A *is denoted by* $R(A)$ *or* $R_k(A)$.

1.2. Our results

It is a natural question to ask which are the algebras whose rank is exactly one larger than the minimum. We say that an algebra has *minimal rank plus one* if

$$R(A) = 2 \dim A - t + 1,$$

where t is the number of maximal twosided ideals in A. We completely solve this question here for semisimple algebras over $\mathbb{R}$. A semisimple $\mathbb{R}$-algebra has minimal rank plus one iff

$$A = \mathbb{H} \times B$$

where B is a semisimple algebra of minimal rank, that is,

$$B = \mathbb{R}^{2\times 2} \times \cdots \times \mathbb{R}^{2\times 2} \times \mathbb{C} \times \cdots \times \mathbb{C} \times \mathbb{R} \times \cdots \times \mathbb{R}.$$

Note that over $\mathbb{R}$, there is only one division algebra of dimension two, namely the complex numbers $\mathbb{C}$ (viewed as an $\mathbb{R}$-algebra), and one division algebra of dimension four, the *Hamiltonian quaternions* $\mathbb{H}$. There are no other nontrivial finite-dimensional division algebras over $\mathbb{R}$. $\mathbb{C}$ is also the only commutative division algebra, that is, extension field over $\mathbb{R}$.

Characterization results as the one that we prove in this paper are important, since they link the algebraic structure of an algebra to its complexity. We can read off the complexity of the algebra from its structure or get at least lower bounds for the complexity by inspecting the algebraic structure.

One result on the way of our characterization is a new lower bound of 17 for $\mathbb{C}^{2\times 2}$ (viewed as an $\mathbb{R}$-algebra). This bound holds for any other extension field of dimension two over arbitrary fields. This new bound improves on the open question posed by Strassen [13, Section 12, Problem 3].

For algebras A with radical of minimal rank plus one, we have two results: If $A/\operatorname{rad} A$ contains one factor $\mathbb{H}$, then $A = \mathbb{H} \times B$ where B is an algebra of minimal rank. If $A/\operatorname{rad} A$ does not contain the factor $\mathbb{H}$, then the situation is not so clear and we do not have a complete algebraic characterization of the algebras of minimal rank plus one. We present some partial results which unfortunately are a little weaker than announced in the proceedings version.

1.3. Outline of the proof

A semisimple algebra A consists of simple factors of the form $D^{n\times n}$ where D is a division algebra. It follows from results by Alder and Strassen that no factor of A can have rank $\geq 2\dim D^{n\times n}+1$ and at least one factor has to have rank $2\dim D^{n\times n}$, i.e., has minimal rank plus one. We show that the only simple algebra that has minimal rank plus one is $\mathbb{H}$, the Hamiltonian quaternions. In particular, we show that $\mathbb{C}^{2\times 2}$ does not have minimal rank plus one in Section 4. (This is the "hardest case".) Next, we show that A cannot have two factors of the form $\mathbb{H}$ in Section 3. With this, we show the characterization result for the semisimple case in Section 5 (Theorem 4). Finally, Section 7 contains the charaterization result of algebras of minimal rank plus one for which $A/\operatorname{rad} A$ has minimal rank plus one and Section 8 contains some results for the case that $A/\operatorname{rad} A$ has minimal rank.

2. Tools for lower bounds

For the reader's convenience, we survey some tools to show lower bounds that we will need in our proof.

2.1. Lower bound techniques

First let us mention some lower bound techniques whose proofs can be found in [9].

Definition 2. *Let U, V, and W be vector spaces over some field k and $\beta = (f_1, g_1, w_1; \dots; f_r, g_r, w_r)$ be a bilinear computation a bilinear map $\phi : U \times V \to W$. Let $U_1 \subseteq U$, $V_1 \subseteq V$, and $W_1 \subseteq W$ be subspaces. We say that the triple (U_1, V_1, W_1) is* separated *by β if there are disjoint sets of indices I, $J \subseteq \{\rho : w_\rho \notin W_1\}$ such that*

$$U_1 \cap \bigcap_{i\in I} \ker f_i = \{0\} \quad \text{and} \quad V_1 \cap \bigcap_{j\in J} \ker g_j = \{0\}.$$

The next lemma will help us find new lower bounds:

Lemma 1. *Let U, V, and W be vector spaces over some field k and let $\beta = (f_1, g_1, w_1; \dots; f_r, g_r, w_r)$ be a bilinear computation for some bilinear map $\phi : U \times V \to W$. Let $U_1 \subseteq U$, $V_1 \subseteq V$, and $W_1 \subseteq W$ be subspaces such that (U_1, V_1, W_1) is separated by β. Let π be an endomorphism of W such that $W_1 \subseteq \ker \pi$. Then we have*

$$r \geq R((\pi \circ \phi)/(U_1 \times V_1)) + \dim U_1 + \dim V_1 + |\{\rho : w_\rho \in W_1\}|.$$

A helpful tool to find "large" triples that are separated by some computation β are the following lemmas due to Alder and Strassen [1]:

Lemma 2 (Extension Lemma). *Let U, V, and W be vector spaces over a field k and β be a bilinear computation for a bilinear map $\phi : U \times V \to W$. Let $U_1 \subseteq U_2 \subseteq U$, $V_1 \subseteq V$, and $W_1 \subseteq W$ be subspaces such that (U_1, V_1, W_1) is separated by β. Then also the triple (U_2, V_1, W_1) is separated by β, or there is some $u \in U_2 - U_1$ such that*

$$\phi(u, V) \subseteq \operatorname{lin} \phi(u, V_1) + W_1.$$

With this tool, Alder and Strassen [1] show the following two lower bounds.

Lemma 3. *Let A be an algebra over a field k. If $A = B \times B'$ with B being a simple k-algebra and B' being an arbitrary k-algebra, then*

$$R(A) \geq 2 \dim B - 1 + R(B').$$

Lemma 4. *Let A be an algebra over some field k. Then*

$$R(A) \geq R(A/\operatorname{rad} A) + 2 \dim \operatorname{rad} A.$$

This lemma also holds if we replace $\operatorname{rad} A$ by any twosided ideal $I \subseteq \operatorname{rad} A$. The last lemma implies the following useful result.

Corollary 1. *If A has minimal rank plus one, then $A/\operatorname{rad} A$ has minimal rank or minimal rank plus one.*

Proof. Let t be the number of maximal twosided ideals of A. It is well known that this is also the number of twosided ideals of $A/\operatorname{rad} A$. We have

$$R(A) = 2 \dim A + t \geq R(A/\operatorname{rad} A) + 2 \dim \operatorname{rad} A. \quad \square$$

Furthermore, we need the following lemma from [4]. For this let $[a, b] := ab - ba$ denote the Lie product of the two elements a, $b \in A$, where A denotes an associative algebra.

Lemma 5. *Let A be an associative algebra over some field k, $\dim A =: N$, and let $\beta = (f_1, g_1, w_1; \ldots; f_r, g_r, w_r)$ be a bilinear computation for A. If 1, a, $b \in \bigcap_{\mu=1}^{m} \ker f_\mu$, then*

$$r \geq m + N + \frac{1}{2} \dim([a, b]A).$$

Exploiting the previous lemma, one can show the following lower bound [4].

Theorem 1. *Let k be a field, D be a finite dimensional k-division algebra and $A = D^{n \times n}$ with $n \geq 2$. Then*

$$R(A) \geq \frac{5}{2} \dim A - 3n.$$

2.2. Equivalence of computations

Often, proofs become simpler when we normalize computations. A simple equivalence transformation of computations is the *permutation* of the products.

Trickier is the so-called *sandwiching*. Let $\beta = (f_1, g_1, w_1; \dots; f_r, g_r, w_r)$ be a computation for an algebra A, i.e.,

$$xy = \sum_{\rho=1}^{r} f_\rho(x) g_\rho(y) w_\rho$$

Let a, b, c be invertible elements of A. Then

$$xy = a(a^{-1}xb)(b^{-1}yc)c^{-1} = \sum_{\rho=1}^{r} f_\rho(a^{-1}xb) g_\rho(b^{-1}yc) a w_\rho c^{-1}.$$

Thus we can replace each f_ρ by $\hat{f}_\rho$ defined by $\hat{f}_\rho(x) = f_\rho(a^{-1}xb)$, g_ρ by $\hat{g}_\rho$ defined by $\hat{g}_\rho(y) = g_\rho(b^{-1}yc)$, and w_ρ by $\hat{w}_\rho = a w_\rho c^{-1}$.

For the next two equivalence transformations, we assume that A is a simple algebra, that is, $A \cong D^{n\times n}$ for some division algebra A. For an element $x \in A$, x^T denotes the transposed of x. Let $\beta = (f_1, g_1, w_1; \dots; f_r, g_r, w_r)$ be a computation for an algebra A. Then

$$y^T x^T = (xy)^T = \sum_{\rho=1}^{r} \tilde{g}_\rho(y^T) \tilde{f}_\rho(x^T) w_\rho^T,$$

where $\tilde{g}_\rho(y)$ is defined by $\tilde{g}_\rho(y) := g_\rho(y^T)$ and $\tilde{f}_\rho(x)$ is defined by $\tilde{f}_\rho(x) := f_\rho(x^T)$. So we can change the f's with the g's (at the cost of *transposing* the w's but this will not do any harm since in our proofs, we usually only care about the rank of the w_ρ and other quantities that are invariant under transposing).

Finally, with every matrix $x \in A$, we can associate a linear form, namely, $y \mapsto \langle x, y\rangle$, where $\langle .,. \rangle$ denotes the standard inner product. (We here view x and y as vectors in $k^{n^2 \cdot \dim D}$.) In this way, we will often identify f_ρ with an element from A, which we abusively call f_ρ again. We have

$$xy = \sum_{\rho=1}^{r} f_\rho(x) g_\rho(y) w_\rho$$

for all $x, y \in A$ iff

$$\langle xy, z\rangle = \sum_{\rho=1}^{r} \langle f_\rho, x\rangle \langle g_\rho, y\rangle \langle w_\rho, z\rangle$$

for all $x, y, z \in A$. Since $\langle xy, z\rangle = \langle xyz^T, 1\rangle = \operatorname{trace}(xyz^T)$ and $\operatorname{trace}(xyz^T) = \operatorname{trace}(z^T xy) = \operatorname{trace}(yz^T x)$, we can *cyclically shift* the f's, g's, and w's in this way. Altogether, the latter two equivalence transformations allow us to permute the f's, g's, and w's in an arbitrary way.

3. A lower bound for $\mathbb{H} \times \mathbb{H}$ over $\mathbb{R}$

In this section, we will prove the the following theorem.

Theorem 2. *We have* $R_{\mathbb{R}}(\mathbb{H} \times \mathbb{H}) = 16$.

Bshouty [7, Cor. 4] has shown that for every division algebra D and every arbitrary algebra A,

$$R(D \times A) \geq 2 \dim D + R(A). \tag{1}$$

This immediatly implies Theorem 2. Bshouty's proof is quite involved; therefore, we provide a shorter proof for Theoren 2, which is of course a weaker statement than (1), below.

PROOF. It is well known that $R_{\mathbb{R}}(\mathbb{H}) = 8$ (this was shown independently by a number of people, see [9]), which implies that $R_{\mathbb{R}}(\mathbb{H} \times \mathbb{H}) \leq 16$. To prove the lower bound, we first will show the following claim:

Claim 1. *If x, $y \in \mathbb{H}$ are such that x, y, and 1 are linearly independent over $\mathbb{R}$, then* $\operatorname{lin}\{1, x, y, x\cdot y\} = \mathbb{H}$.

Let x, $y \in \mathbb{H}$ have the above mentioned properties. The inner automorphisms act on $\mathbb{H}$ via rotation in $\mathbb{R}^3$ on the last three coordinates of each quaternion. Hence, we can assume w.l.o.g that $x = x_1\cdot 1 + x_2\cdot i$ and $y = y_1\cdot 1 + y_2\cdot i + y_3\cdot j$, x_ν, $y_\nu \in \mathbb{R}$. Since 1, x, and y are still linearly independent, we know that $x_2 \neq 0 \neq y_3$ and hence $\operatorname{lin}\{1, x, y\} = \operatorname{lin}\{1, i, j\}$. Furthermore, the last coordinate of $x\cdot y$ equals $x_2 y_3$ and is hence not equal to zero, which proves the claim.

Let $\beta = (f_1, g_1, w_1; \ldots; f_r, g_r, w_r)$ be a computation for $\mathbb{H} \times \mathbb{H}$. We can choose two elements $\hat{a} = (a, a')$ and $\hat{b} = (b, b') \in \mathbb{H} \times \mathbb{H}$ such that their span is contained in the intersection of at least six of the kernels of $f_1, \ldots, f_r$ and where a and b are linearly independent vectors in $\mathbb{R}^4$. W.l.o.g., assume that $\operatorname{lin}\{\hat{a}, \hat{b}\} \subseteq \ker f_1 \cap \cdots \cap \ker f_6$.

If for all possible choices $a' = 0$ and $b' = 0$, then we can split the computation into two separate computations for $\mathbb{H}$ and get a lower bound of $8 + 8 = 16$. Thus we can assume that $a' \neq 0$. Via sandwiching, we can achieve that $a = 1$ and furthermore, by letting inner automorphisms act, that $b \in \operatorname{lin}\{1, i\}$. Since $a' \neq 0$, it follows that $g_7, \ldots, g_r$ generate $(\mathbb{H} \times \mathbb{H})^*$. Now, choose a vector $\hat{c} = (c, c')$, $c \neq 0$, that is contained in the intersection of the kernels of at least seven of the vectors $g_7, \ldots, g_r$ and use sandwiching to achieve $c = 1$. W.l.o.g., let $\hat{c}$ be contained in $\ker g_7 \cap \cdots \cap \ker g_{13}$. Finally, we can choose an element $\hat{d} = (d, d')$ in the intersection of the kernels of at least six of $g_7, \ldots, g_{13}$ such that 1, b, and d are linearly independent over $\mathbb{R}$. W.l.o.g., assume that $\operatorname{lin}\{\hat{c}, \hat{d}\} \subseteq \ker g_7 \cap \cdots \cap \ker g_{12}$. The above claim shows that $a\cdot c = 1$, $a\cdot d = d$, $b\cdot c = b$, and $b\cdot d$ span $\mathbb{H}$, which yields that the products $\hat{a}\cdot\hat{c}$, $\hat{a}\cdot\hat{d}$, $\hat{b}\cdot\hat{c}$, and $\hat{b}\cdot\hat{d}$ span a four dimensional vector space over $\mathbb{R}$. On the other hand, we know that by construction, each of these products lies in the span of $\operatorname{lin}\{w_{13}, \ldots, w_r\}$. Hence, r has to be at least 16. $\square$

4. A lower bound for $\mathbb{C}^{2\times 2}$ over $\mathbb{R}$

The goal of this section is to prove the following theorem.

Theorem 3. *We have* $R_{\mathbb{R}}(\mathbb{C}^{2\times 2}) \geq 17$.

We will prove this theorem in two steps. We define the following property for computations. A computation $\beta = (f_1, g_1, w_1; \dots; f_r, g_r, w_r)$ has the property (*) if the following holds:

(*) If there is a matrix $x \in \mathbb{C}^{2\times 2} \setminus \{0\}$ for which there exist three different indices ν_1, ν_2, and $\nu_3 \in \{1, \dots, r\}$ such that

$$\begin{aligned} \mathrm{lin}\{x, i{\cdot}x\} &\subseteq \ker f_{\nu_1} \cap \ker f_{\nu_2} \cap \ker f_{\nu_3} \text{ or} \\ \mathrm{lin}\{x, i{\cdot}x\} &\subseteq \ker g_{\nu_1} \cap \ker g_{\nu_2} \cap \ker g_{\nu_3} \text{ or} \\ \mathrm{lin}\{x, i{\cdot}x\} &\subseteq \mathrm{lin}\{w_{\nu_1}, w_{\nu_2}, w_{\nu_3}\}^{\perp}, \end{aligned}$$

then x is of rank two, where $V^{\perp}$ is the space of all vectors u that fulfill $\langle v, u\rangle = 0$ for all $v \in V$ and $\langle ., . \rangle$ denotes the standard inner product.

In Subsection 4.1 we show that a computation for $\mathbb{C}^{2\times 2}$ of length 16 must satisfy (*) and in Subsection 4.2 we show that no such computation exists.

4.1. Computations not satisfying property (*)

If a computation does not satisfy (*), then—with some refinements—standard methods work, as we will see in this section. Problematic are computations that do satisfy (*). We will deal with these in the next section.

For a field k, let $\langle e, h, l\rangle_k$ denote the matrix multiplication tensor of dimensions $e \times h$, $h \times l$, and $e \times l$ having coefficients in k.

Lemma 6. $R_{\mathbb{R}}(\langle 1, 1, 2\rangle_{\mathbb{C}}) = 6$.

Proof. This tensor has rank at most six, since the complex multiplication has rank three over $\mathbb{R}$. Assume that there exists a computation

$$(f_1, g_1, w_2; \dots; f_5, g_5, w_5)$$

of length five for $\langle 1, 1, 2\rangle$. Then we can (possibly after permuting the products) assume that f_1, f_2 are a basis of $\mathbb{C}^*$ and that $g_2, \dots, g_5$ form a basis of $(\mathbb{C}^{1\times 2})^*$. Let x_1, x_2 and $y_1, \dots, y_4$ be the bases dual to f_1, f_2 and $g_2, \dots, g_5$, respectively. Then we can choose an index $\nu \in \{2, \dots, 4\}$ such that y_1 and y_ν are linearly independent over $\mathbb{C}$, which means that the span of $\mathrm{lin}\{x_1y_1, x_1y_\nu, x_2y_1, x_2y_\nu\}$ is a four dimensional vector space over $\mathbb{R}$. But for $i \in \{1, 2\}$ and $j \in \{1, \nu\}$, $x_iy_j \in \mathrm{lin}\{w_1, w_2, w_\nu\}$. Since the latter is a vector space over $\mathbb{R}$ with dimension at most three, we get a contradiction. □

Lemma 7. *Let u, v, and $w \in \mathbb{C}^{2\times 2}$ and assume that there exists a rank one matrix x such that $\mathrm{lin}\{x, ix\} \subseteq \mathrm{lin}\{u, v, w\}^{\perp}$ over $\mathbb{R}$. Then we can find invertible matrices a and b such that $(aub)_{11} = (avb)_{11} = (awb)_{11} = 0$, where $(\, . \,)_{11}$ denotes the entry in position $(1, 1)$.*

PROOF. Let $x = (x_{11}, x_{12}, x_{21}, x_{22})$, $x_{\nu\mu} = (x'_{\nu\mu}, x''_{\nu\mu}) \in \mathbb{C}$, be a matrix with the above property. (To save some space, we write matrices occasionally as column vectors.) Let z be any of the vectors u, v, or w. The vectors $-ix$ (for convenience) and x being perpendicular to $z = (z_{11}, z_{12}, z_{21}, z_{22})$, $z_{\nu\mu} = (z'_{\nu\mu}, z''_{\nu\mu}) \in \mathbb{C}$, means that we have

$$\sum_{\nu,\mu=1}^{2} \begin{pmatrix} x''_{\nu\mu} & -x'_{\nu\mu} \\ x'_{\nu\mu} & x''_{\nu\mu} \end{pmatrix} \begin{pmatrix} z'_{\nu\mu} \\ z''_{\nu\mu} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since the matrix $\begin{pmatrix} x''_{\nu\mu} & -x'_{\nu\mu} \\ x'_{\nu\mu} & x''_{\nu\mu} \end{pmatrix}$ is the left multiplication matrix of $\hat{x}_{\nu\mu} = i\cdot\bar{x}_{\nu\mu}$, we can also write the above sum as $\sum_{\nu,\mu=1}^{2} \hat{x}_{\nu\mu}\cdot z_{\nu\mu} = 0$. Note that the matrix $\hat{x} := (\hat{x}_{11}, \hat{x}_{12}, \hat{x}_{21}, \hat{x}_{22})$ with $\hat{x}_{\nu\mu} := i\cdot\bar{x}_{\nu\mu} = (x''_{\nu\mu}, x'_{\nu\mu})$ has rank one, too. On the other hand, multiplying z from the left by $a = (a_{11}, a_{12}, a_{21}, a_{22})$ and from the right by $b = (b_{11}, b_{12}, b_{21}, b_{22})$ yields $(azb)_{11} = a_{11}b_{11}z_{11} + a_{11}b_{21}z_{12} + a_{12}b_{11}z_{21} + a_{12}b_{21}z_{22}$. If we find a_{11}, a_{12}, b_{11}, $b_{21} \in \mathbb{C}$ such that $a_{11}b_{11} = \hat{x}_{11}$, $a_{11}b_{21} = \hat{x}_{12}$, $a_{12}b_{11} = \hat{x}_{21}$, and $a_{12}b_{21} = \hat{x}_{22}$, then we are done. This is equivalent to finding two 2-dimensional vectors (a_{11}, a_{12}) and (b_{11}, b_{21}) with complex entries such that

$$(a_{11}, a_{12}) \otimes (b_{11}, b_{21}) = \begin{pmatrix} \hat{x}_{11} & \hat{x}_{12} \\ \hat{x}_{21} & \hat{x}_{22} \end{pmatrix}.$$

This is possible if and only if $\hat{x}$ has rank one, which had been one of our assumptions. Furthermore, since $\hat{x} \neq 0$, neither both a_{11} and a_{12} nor both b_{11} and b_{21} can be zero. Hence, we can construct invertible matrices $(a_{11}, a_{12}, a_{21}, a_{22})$ and $(b_{11}, b_{12}, b_{21}, b_{22})$ such that $(azb)_{11} = 0$ for all $z \in \{u, v, w\}$. $\square$

Proposition 1. *Let $\beta := (f_1, g_1, w_1; \ldots; f_r, g_r, w_r)$ be a computation that does not satisfy (*). Then $r \geq 17$.*

PROOF. Since β does not satisfy (*), we can find three indices ν_1, ν_2, $\nu_3 \in \{1, \ldots, r\}$ and a rank one matrix x such that $\mathrm{lin}\{x, i\cdot x\} \subseteq \ker f_{\nu_1} \cap \ker f_{\nu_2} \cap \ker f_{\nu_3}$, $\mathrm{lin}\{x, i\cdot x\} \subseteq \ker g_{\nu_1} \cap \ker g_{\nu_2} \cap \ker g_{\nu_3}$, or $\mathrm{lin}\{x, i\cdot x\} \subseteq \mathrm{lin}\{w_{\nu_1}, w_{\nu_2}, w_{\nu_3}\}^{\perp}$. W.l.o.g., assume that $\nu_1 = 1$, $\nu_2 = 2$, and $\nu_3 = 3$ and that $\mathrm{lin}\{x, i\cdot x\} \subseteq \mathrm{lin}\{w_1, w_2, w_2\}^{\perp}$, for otherwise, we could exchange the f's or g's with the w's.[1] Then, by Lemma 7, we can achieve (via sandwiching) that

$$W := \mathrm{lin}\{w_1, w_2, w_3\} \subseteq \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}.$$

Define the two left and two right ideals L_1, L_2, R_1, and R_2 as follows:

$$L_1 := \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix}, \quad L_2 := \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}, \quad R_1 := \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}, \quad R_2 := \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix}.$$

[1] Strictly speaking, we can only exchange the adjoints of the f's and g's with the w's, see Section 2.2. But since "having rank one" is invariant under transposing, this does not matter.

Each ideal is a four dimensional vector space over $\mathbb{R}$. For the following claims, define the computation $\beta' := (\tilde{g}_1, \tilde{f}_1, w_1^T; \dots; \tilde{g}_r, \tilde{f}_r, w_r^T)$, that is obtained by transposing as described in Section 2.2.

Claim 2. *The triple* $(\{0\}, L_2, W)$ *is separable by* β *and the triple* $(\{0\}, L_2, W^T)$ *is separable by* β'*, where* $W^T := \{w^T : w \in W\}$.

Assume $(\{0\}, L_2, W)$ is not separable by β. By the Extension Lemma (Lemma 2), there exists an element $l \in L_2 \setminus \{0\}$ such that $\mathbb{C}^{2\times 2} \cdot l \subseteq \{0\} \cdot l + W = W$. But $\mathbb{C}^{2\times 2} \cdot l = L_2$ is four dimensional (over $\mathbb{R}$), whereas W has dimension at most three. The second part of the claim is shown in a similar fashion.

Claim 3. *The triple* (R_2, L_2, W) *is separable by* β *or the triple* (R_2, L_2, W^T) *is separable by* β'.

Assume (R_2, L_2, W) is not separable by β. By the Extension Lemma, there exists an element $r \in R_2 \setminus \{0\}$ such that $r \cdot \mathbb{C}^{2\times 2} \subseteq R_2 \cdot L_2 + W$. Now, $r \cdot \mathbb{C}^{2\times 2} = R_2$ and $R_2 \cdot L_2$ contains exactly all matrices with a nonzero entry only in the lower right corner. We distinguish three different cases:

1. $\dim(W + R_2) \geq 6$:[2] Then the image of the projection
$$\pi_{12} : W \to \mathbb{C}, \quad \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \mapsto b,$$
is two dimensional and hence, the space $W \cap R_2$ is at most one dimensional. Furthermore, $R_2 \cdot L_2$ is two dimensional. Thus the four dimensional space R_2 cannot be contained in $R_2 \cdot L_2 + W$.
2. $\dim(W + L_2) \geq 6$: In this case, we can use the computation β'. But then from $\dim(W + L_2) \geq 6$ it follows that $\dim(W^T + R_2) \geq 6$ and hence, by case (1), (R_2, L_2, W^T) is separable by β'.
3. $\dim(W + L_2) \leq 5$: Then the image of the projection
$$\pi_{21} : \quad W \to \mathbb{C}, \qquad \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \mapsto c,$$
is at most one dimensional, which shows that the whole ideal R_2 cannot lie in the space $R_2 \cdot L_2 + W$. This proves Claim 3.

W.l.o.g. assume that (R_2, L_2, W) is separable by β and define the projection
$$\pi : \quad \mathbb{C}^{2\times 2} \longrightarrow \mathbb{C}^{2\times 2}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Let ϕ be the multiplication of $\mathbb{C}^{2\times 2}$. Since $W \subseteq \ker \pi$, it follows that
$$R(\pi \circ \phi / R_2 \times L_2) + \dim(R_2 \times L_2) + \dim W \leq r$$

[2] Note that because of the special structure of W, we even have $\dim(W + R_2) = 6$.

by Lemma 1. Hence $R(\pi \circ \phi/R_2 \times L_2) + 11 \leq r$. Now, the bilinear map $\pi \circ \phi/R_2 \times L_2$ is a map

$$\pi \circ \phi/R_2 \times L_2 : \mathbb{C}^{2\times 2}/R_2 \times \mathbb{C}^{2\times 2}/L_2 \to \mathbb{C}^{2\times 2}/\ker \pi.$$

But $\mathbb{C}^{2\times 2}/R_2 = R_1$, $\mathbb{C}^{2\times 2}/L_2 = L_1$, and $\mathbb{C}^{2\times 2}/\ker \pi = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}$. It follows that $\pi \circ \phi/R_2 \times L_2 \cong \langle 1, 2, 1\rangle$, the complex matrix multiplication tensor $\langle 1, 2, 1\rangle$ over $\mathbb{R}$. By Lemma 6, the tensor $\langle 1, 1, 2\rangle$ has rank six. Since this tensor is isomorphic to the tensor $\langle 1, 2, 1\rangle$, we get

$$r \geq R(\pi \circ \phi/R_2 \times L_2) + 11 = 6 + 11 = 17. \quad \Box$$

4.2. Computations satisfying property (*)

Lemma 8. *Let $\beta := (f_1, g_1, w_1; \ldots ; f_{16}, g_{16}, w_{16})$ satisfy (*). Then we can achieve (possibly after permutation), that $f_1, \ldots, f_8$ and $w_9, \ldots, w_{16}$ are bases of $\mathbb{C}^{2\times 2}$.*

Proof. We can assume that $f_1, \ldots, f_8$ is a basis. We can also assume that $g_9, \ldots, g_{16}$ and $w_9, \ldots, w_{16}$ are linearly dependent, respectively. Otherwise, after probably exchanging the g's and w's, we are finished. Then the following claim holds:

Claim 4. *$g_1, \ldots, g_8$ and $w_1, \ldots, w_8$ are bases of $\mathbb{C}^{2\times 2}$ and for all $\nu \in \{1, \ldots, 8\}$,* $\dim \mathrm{lin}\{g_9, \ldots, g_{16}, g_\nu\} = \dim \mathrm{lin}\{w_9, \ldots, w_{16}, w_\nu\} = 8$.

Exchanging the f's and w's (again we can skip the adjoints here) gives a computation $\beta' := (w_1, g_1, f_1; \ldots ; w_{16}, g_{16}, f_{16})$ for the same tensor. Assume that a nonzero matrix $y \in \ker g_9 \cap \cdots \cap \ker g_{16}$ has rank one. There is a rank one matrix x such that $x\cdot y = 0 = ix\cdot y$. But this means that $x\cdot y = \sum_{\nu=1}^{8} w_\nu(x) g_\nu(y) f_\nu = 0$ and $ix\cdot y = \sum_{\nu=1}^{8} w_\nu(ix) g_\nu(y) f_\nu = 0$. Since $f_1, \ldots, f_8$ are linearly independent, we get $w_\nu(x) g_\nu(y) = w_\nu(ix) g_\nu(y) = 0$ for $\nu \in \{1, \ldots, 8\}$. Now, the image of R_y, the right multiplication with y, is four dimensional, hence, at least four of $g_\nu(y)$, $1 \leq \nu \leq 8$, are nonzero. But then at least for four indices $1 \leq \nu \leq 8$ we have $w_\nu(x) = w_\nu(ix) = 0$, which is a contradiction to property (*). This means that the matrix y has rank two and so the image of $R_y = \sum_{\nu=1}^{8} g_\nu(y) w_\nu \otimes f_\nu$ is full dimensional. On the one hand, this implies that $w_1, \ldots, w_8$ has to be a basis. On the other hand, we see that $g_\nu(y)$ has to be nonzero for all $\nu \leq 8$, which proves the second part of the claim. (Note that $\dim \mathrm{lin}\{g_9, \ldots, g_{16}\} \geq 7$, since otherwise, we could find an invertible matrix in $\ker g_8 \cap \cdots \cap \ker g_{16}$ with the same arguments as above, which is a contradiction.) Similarly, after exchanging the g's and w's, one can conclude the same assertions for the g's. This proves the claim.

Showing that there exists a partition $I, J \subseteq \{1, \ldots, 16\}$ such that $|I| = |J| = 8$ and $\{f_i : i \in I\}$ and $\{w_j : j \in J\}$ are both bases would prove the lemma.

Now, the claim above shows that if we choose an index set $J' \subset \{9, \ldots, 16\}$, $|J'| = 7$, such that $\{g_j : j \in J'\}$ are linearly independent, every g_ν, $\nu \leq 8$, would lead to a basis $\{g_j : j \in J_\nu\}$, where $J_\nu := J' \cup \{\nu\}$. Let μ be such that $\{\mu\} = \{9, \dots, 16\} - J'$. Then, by Steinitz exchange, there has to be a $\nu \in \{1, \dots, 8\}$ such that

$$\{w_j : j \in (\{1, \dots, 8\} - \{\nu\}) \cup \{\mu\}\}$$

is a basis. Exchanging the g's and the f's and setting $I := (\{1, \dots, 8\} - \{\nu\}) \cup \{\mu\}$ and $J := J_\nu$ gives a partition with the desired properties. □

Lemma 9. *Let $x_1, \dots, x_5 \in \mathbb{C}^{2\times 2}$ be five matrices that are linearly independent over $\mathbb{R}$. Then $\mathrm{lin}\{x_1, \dots, x_5\}$ contains a matrix of rank two.*

Proof. We can assume that x_j, $1 \le j \le 5$, have rank one, otherwise the assertion is trivial. This means that there exist vectors a_j, $b_j \in \mathbb{C}^2$ such that $x_j = a_j \otimes b_j$ for $1 \le j \le 5$. Hence, the elements of $\mathrm{lin}\{x_1, \dots, x_5\}$ are exactly of the form $\sum_{j=1}^{5} \lambda_j a_j \otimes b_j$, $\lambda_j \in \mathbb{R}$. If every such element has rank one, then we must have

$$\dim_{\mathbb{C}} \mathrm{lin}\{a_1, \dots, a_5\} = 1 \quad \vee \quad \dim_{\mathbb{C}} \mathrm{lin}\{b_1, \dots, b_5\} = 1.$$

But then $\dim_{\mathbb{C}} \mathrm{lin}\{a_1 \otimes b_1, \dots, a_5 \otimes b_5\} \le 2$ and hence $\dim_{\mathbb{R}} \mathrm{lin}\{a_1 \otimes b_1, \dots, a_5 \otimes b_5\} \le 4$. This is a contradiction to the assumption that the matrices are linearly independent over $\mathbb{R}$. □

Lemma 10. *Let $U \subseteq \mathbb{C}^{2\times 2}$ be a three dimensional subspace over $\mathbb{R}$ of rank one matrices. Then there exists an $x \in U$ such that $ix \in U$.*

Proof. Let $U = \mathrm{lin}\{y_1, y_2, y_3\}$. By Lemma 9, the span of five matrices that are linearly independent over $\mathbb{R}$ already contains an invertible matrix. Hence, $\dim_{\mathbb{R}} \mathrm{lin}\{y_1, y_2, y_3, iy_1, iy_2, iy_3\} \le 4$. Thus there exists a matrix y_4 such that $iy_j \in \mathrm{lin}\{y_1, y_2, y_3, y_4\}$ for all $j \le 4$. Let $iy_1 = \sum_{j=1}^{4} \lambda_j y_j$ and $iy_2 = \sum_{j=1}^{4} \mu_j y_j$. If $\lambda_4 = 0$ we can choose $x = y_1$. Otherwise, we define $x := iy_2 - \frac{\mu_4}{\lambda_4} iy_1$. □

Proposition 2. *There does not exist any computation for $\mathbb{C}^{2\times 2}$ over $\mathbb{R}$ of length 16 that satisfies (*).*

Proof. Assume there exists such a computation

$$\beta := (f_1, g_1, w_1; \dots; f_{16}, g_{16}, w_{16})$$

that satisfies (*). By Lemma 8, we can assume that $f_1, \dots, f_8$ and $w_9, \dots, w_{16}$ are bases. Let $x_1, \dots, x_8$ be the basis dual to $f_1, \dots, f_8$.

Claim 5. *For each $j \le 8$, the rank of x_j is two.*

Assume that the rank of x_j is one. Since the rank of L_{x_j}, the 8×8-matrix induced by the left multiplication with x_j, is four, there are four matrices $y_1, \dots, y_4$

that are linearly independent over $\mathbb{R}$ such that $x_j \cdot y_k = 0$ for all $k \leq 4$. Define the subspace $U := \mathrm{lin}\{y_1, \ldots, y_4\} \cap \ker g_j$. For each $y \in U$ we then have

$$x_j \cdot y = \sum_{\nu=1}^{16} f_\nu(x_j) g_\nu(y) w_\nu = \sum_{\nu=9}^{16} f_\nu(x_j) g_\nu(y) w_\nu = 0.$$

But $w_9, \ldots, w_{16}$ is a basis. So $(f_9(x_j)g_9(y), \ldots, f_{16}(x_j)g_{16}(y))$ must be the zero vector. Since the rank of L_{x_j} is four, at least three of the $f_\nu(x_j)$, $\nu \geq 9$, are nonzero. This means that at least for three indices $\nu \geq 9$, we have $g_\nu(y) = 0$ for every $y \in U$. But U is at least three dimensional and contains only rank one matrices. Hence, Lemma 10 tells us that we can find a vector $x \in U$ such that $ix \in U$. Now x has rank one and $\mathrm{lin}\{x, ix\}$ is contained in the intersection of at least three $\ker g_\nu$, which contradicts property (*) and hence proves the claim.

This shows that via sandwiching we can achieve that $x_1 = 1$ is the unit matrix and x_2 is in Jordan normal form. We consider three different cases depending on the Jordan normal form of x_2.

1. x_2 has two different eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$: In this case, we use Lemma 5. For this, note that since

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} - \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & (\lambda_1 - \lambda_2)x_{12} \\ (\lambda_2 - \lambda_1)x_{21} & 0 \end{pmatrix},$$

$[x_2, x]$ is invertible if $x_{12} \neq 0 \neq x_{21}$.

Claim 6. *There is an index $\nu \in \{3, \ldots, 8\}$ such that $[x_2, x_\nu]$ is invertible.*

Assume that none of the matrices $x_3, \ldots, x_8$ fulfills this property, i.e., that either $(x_\nu)_{12}$ or $(x_\nu)_{21}$ is zero. Then we can find at least three matrices x_{ν_1}, x_{ν_2}, x_{ν_3}, $\nu_j \geq 3$, such that $(x_{\nu_1})_{12} = (x_{\nu_2})_{12} = (x_{\nu_3})_{12} = 0$ or $(x_{\nu_1})_{21} = (x_{\nu_2})_{21} = (x_{\nu_3})_{21} = 0$. W.l.o.g., assume that we are in the first case and that $\nu_1 = 3$, $\nu_2 = 4$, and $\nu_3 = 5$. Then consider the space U defined by

$$U := \mathrm{lin}\{x_1, \ldots, x_5\} \cap \mathrm{lin}\{\begin{pmatrix} (1,0) & (0,0) \\ (0,0) & (0,0) \end{pmatrix}, \begin{pmatrix} (0,1) & (0,0) \\ (0,0) & (0,0) \end{pmatrix}\}^\perp.$$

Since $\mathrm{lin}\{x_1, \ldots, x_5\}$ is five dimensional, the dimension of U is at least three. Furthermore, U contains only matrices where the entries in the first row are zero, i.e., only matrices of rank one. By Lemma 10, U contains a rank one matrix x such that $ix \in U$. But, by construction, x and ix are in $\ker f_6 \cap \ker f_7 \cap \ker f_8$, which is a contradiction to property (*).

W.l.o.g. let x_3 be such that $[x_2, x_3]$ is invertible. Then, choosing $m = 8 - 3 = 5$ in Theorem 5, we get that the length of the computation is at least

$$m + 8 + \frac{1}{2}\dim([x_2, x_3]\mathbb{C}^{2\times 2}) = 5 + 8 + 4 = 17.$$

2. x_2 has the same eigenvalue λ twice and a nilpotent part:
   This means, x_2 is of the form $x_2 = \lambda I_2 + n$, where n is the matrix that has a one in the upper right corner and zeros elsewhere. But for any matrix x we then have

$$\begin{aligned}[x_2, x] &= [n, x] \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} - \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} x_{21} & x_{22} - x_{11} \\ 0 & -x_{21} \end{pmatrix},\end{aligned}$$

   which is an invertible matrix if $x_{21} \neq 0$. Since $x_1, \ldots, x_8$ is a basis, we can find an index $\nu \in \{3, \ldots, 8\}$, such that $(x_\nu)_{21} \neq 0$. W.l.o.g. let $\nu = 3$ be such an index. Then $[x_2, x_3]$ is invertible and by Theorem 5, we get that the computation must have length at least 17, as in the first case.
3. x_2 has the same eigenvalue twice without a nilpotent part:
   Then, since x_2 is also invertible and linearly independent from x_1, we know that $\text{lin}\{x_1, x_2\} = \text{lin}\{1, i\cdot 1\}$. Since L_{x_1} is invertible, we know that $g_1, g_9, \ldots, g_{16}$ generate $\mathbb{C}^{2\times 2}$ as an $\mathbb{R}$-vector space. Hence, we can choose indices $\nu_1 \ldots, \nu_8 \in \{1, 9, \ldots, 16\}$ such that $g_{\nu_1}, \ldots, g_{\nu_8}$ is a basis. Let $y_1, \ldots, y_8$ be the corresponding dual basis. W.l.o.g. we can assume that $y_1, \ldots, y_4$ generate $\mathbb{C}^{2\times 2}$ as a $\mathbb{C}$-vector space. This means that

$$\text{lin}\{x_i y_j : 1 \leq i \leq 2, 1 \leq j \leq 4\} = \mathbb{C}^{2\times 2} \tag{2}$$

   over $\mathbb{R}$. On the other hand, we have

$$x_i y_j \quad = g_i(y_j) w_i \quad + \sum_{\mu=1}^{4} f_{\nu_\mu}(x_i) g_{\nu_\mu}(y_j) w_{\nu_\mu} + f_l(x_i) g_l(y_j) w_l,$$

   where l is defined by $\{l\} = \{1, 9, \ldots, 16\} - \{\nu_1, \ldots, \nu_8\}$, and hence

$$x_i y_j \in \text{lin}\{w_1, w_2, w_{\nu_1}, w_{\nu_2}, w_{\nu_3}, w_{\nu_3}, w_l\}$$

   for $1 \leq i \leq 2$ and $1 \leq j \leq 4$. But the latter is a vector space of dimension at most seven, which is a contradiction to (2). □

5. Semisimple algebras of minimal rank plus one

Theorem 4. *Let A be a semisimple algebra over $\mathbb{R}$ of rank $2\dim A - t + 1$, where t is number of maximal twosided ideals of A. Then A is of the form*

$$A \cong \mathbb{H} \times \mathbb{R}^{2\times 2} \times \cdots \times \mathbb{R}^{2\times 2} \times \mathbb{C} \times \cdots \times \mathbb{C} \times \mathbb{R} \times \cdots \times \mathbb{R}.$$

Proof. Let A be a semisimple $\mathbb{R}$-algebra. By Wedderburn's Theorem, we know that A is isomorphic to an algebra $A_1 \times \cdots \times A_t$, A_τ simple, i.e., $A_\tau \cong D_\tau^{n_\tau \times n_\tau}$,

D_τ a division algebra over $\mathbb{R}$. By [9, Lemma 17.23] and using induction, we obtain

$$R(A) \geq 2\dim A - t - (2\dim A_\tau - 1) + R(A_\tau).$$

Since A is supposed to have rank $2\dim A - t + 1$, we see that

$$R(A_\tau) \leq R(A) - 2\dim A + t + (2\dim A_\tau - 1) = 2\dim A_\tau. \tag{3}$$

Hence, by Theorem 1, no factor A_τ can be a matrix algebra of the form $D_\tau^{n_\tau \times n_\tau}$ with $n_\tau \geq 3$ and $\dim D_\tau \geq 2$. The case $n_\tau \geq 3$ and $\dim D_\tau = 1$, i.e., $D_\tau = \mathbb{R}$, is also not possible. This follows from the lower bound for matrix multiplication in [5]. Now consider $A_\tau = D_\tau^{2\times 2}$. The case $\dim D_\tau \geq 4$, that is, $D = \mathbb{H}$ is not possible, since Theorem 1 tells us that

$$R(A_\tau) \geq \frac{5}{2}\dim A_\tau - 6 = 10\dim_{\mathbb{R}}(D_\tau) - 6,$$

which is greater than $2\dim A_\tau$, since $\dim D_\tau \geq 4$. Because of (3), this also excludes algebras of the above form from being a factor of A. Furthermore, there is no real division algebra of dimension three and Theorem 3 shows that also $\mathbb{C}^{2\times 2}$ cannot be one of the factors.

This shows that the only factors can be $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^{2\times 2}$, and $\mathbb{H}$. From these factors, only the latter one is an algebra that is not of minimal rank, hence it must be contained in A at least once. On the other hand, from Theorem 2 it follows that

$$R_{\mathbb{R}}(\mathbb{H} \times \mathbb{H}) = 16 > 2\dim(\mathbb{H} \times \mathbb{H}) - 1,$$

which shows that $\mathbb{H} \times \mathbb{H}$ cannot be a factor of A. □

6. Algebras with radical

Next, we turn to algebras with radical. We will show two partial results. We need some structural properties of algebras which are shown in [6]. For the reader's convenience, we will briefly repeat the neccessary results in this section.

Since $\mathbb{R}$ is a perfect field, we can apply the Wedderburn–Malcev Theorem. It states that if A is a finite dimensional algebra over a perfect field k, then there exists a semisimple subalgebra B of A such that $B \oplus (\operatorname{rad} A) = A$ and $B \cong A/\operatorname{rad} A$. (The term "subalgebra" here includes that A and B share the same unit element.)

For the rest of this work, A is a finite dimensional $\mathbb{R}$-algebra, and B is a subalgebra of A that fulfils the assertion of the Wedderburn–Malcev Theorem, that is, $B \oplus (\operatorname{rad} A) = A$ and $B \cong A/\operatorname{rad} A$. Since B is semisimple, it is isomorphic to a finite product $B_1 \times \cdots \times B_t$ of simple algebras B_τ by Wedderburn's Theorem. Let $i : B \to B_1 \times \cdots \times B_t$ be an isomorphism of algebras. Since $B_\tau \cong i^{-1}(\{0\} \times \cdots \times \{0\} \times B_\tau \times \{0\} \times \cdots \times \{0\})$, we may view B_τ via i^{-1} as a subspace of B, which only fails to be a subalgebra of B because it has a different unit element. In this sense, we will write the decomposition $B = B_1 \oplus \cdots \oplus B_t$ in an additive way and look at the B_τ as subspaces of B.

This is done to simplify notations, mostly to write B_τ instead of the clumsy $\{0\} \times \cdots \times \{0\} \times B_\tau \times \{0\} \times \cdots \times \{0\}$. It is nevertheless helpful to keep the direct product form of the decomposition in mind, specifically that $B_\tau \cdot B_\sigma = \{0\}$ for $\tau \neq \sigma$.

By Lemma 4, if A has minimal rank plus one, then $B \cong A/\operatorname{rad} A$ either has minimal rank or minimal rank plus one (cf. Corollary 1). But B is a semisimple $\mathbb{R}$-algebra, so we know the structure of B in both cases.

Note that the algebras $B_1, \dots, B_t$ are idempotent, i.e., $B_\tau^2 = B_\tau$ and mutually orthogonal, i.e, $B_\sigma \cdot B_\tau = \{0\}$ for $\sigma \neq \tau$. This implies that for any vector space $V \subseteq A$, $(B_\sigma \cdot V) \cap (B_\tau \cdot V) = \{0\}$ and $(V \cdot B_\sigma) \cap (V \cdot B_\tau) = \{0\}$ for $\sigma \neq \tau$. In particular, we have the decomposition

$$A = \bigoplus_{1 \le \sigma, \tau \le t} B_\sigma \cdot A \cdot B_\tau \qquad \text{and} \qquad \operatorname{rad} A = \bigoplus_{1 \le \sigma, \tau \le t} B_\sigma (\operatorname{rad} A) B_\tau.$$

The following lemma is shown in [6].

Lemma 11. *Let B be a subalgebra of the algebra A with $A = B \oplus \operatorname{rad} A$ and $B \cong A/\operatorname{rad} A$. Let $B = B_1 \oplus \cdots \oplus B_t$ with simple B_τ. Assume that $B_\tau(\operatorname{rad} A)B_1 = B_1(\operatorname{rad} A)B_\tau = \{0\}$ for all $2 \le \tau \le t$. Then for $B' = B_2 + \cdots + B_t$, we have that*

$$A \cong (B_1 + B_1(\operatorname{rad} A)B_1) \times (B' + B'(\operatorname{rad} A)B').$$

7. B has minimal rank plus one

If B has minimal rank plus one, then w.l.o.g. $B_1 = \mathbb{H}$. We will show that in this case $A \cong \mathbb{H} \times A'$ with $A'/\operatorname{rad} A' = B_2 \oplus \cdots \oplus B_t$. Then we will show that A' has minimal rank. Altogether, we will obtain the following result.

Theorem 5. *Let A be an algebra of minimal rank plus one over $\mathbb{R}$ such that $A/\operatorname{rad} A$ has minimal rank plus one. Then $A = \mathbb{H} \times A'$ for some algebra A' of minimal rank.*

7.1. Decomposition

We will show that if there is a $\tau \geq 2$ such that $B_1(\operatorname{rad} A)B_\tau \neq \{0\}$ or $B_\tau(\operatorname{rad} A)B_1 \neq \{0\}$, then A does not have minimal rank plus one. This means that we can apply Lemma 11. In Section 7.3, we will show that $B_1(\operatorname{rad} A)B_1 = \{0\}$, which completes the proof of Theorem 5.

We can restrict ourselves to algebras with $(\operatorname{rad} A)^2 = \{0\}$ as the following lemma shows. This lemma is shown in [6] for algebras of minimal rank.

Lemma 12. *Let A be an algebra as in Theorem 5. Let B be a subalgebra of A with $A = B \oplus \operatorname{rad} A$ and $B \cong A/\operatorname{rad} A$. Let $B = B_1 \oplus \cdots \oplus B_t$ with simple algebras B_τ. Let $\bar{A}$, $\bar{B}$, and $\bar{B}_\tau$ denote the images of A, B, and B_τ under the canonical projection $A \to A/(\operatorname{rad} A)^2$. Then $\bar{A}$ has minimal rank plus one. Furthermore, if $\bar{B}_1 \cdot (\operatorname{rad} \bar{A}) = \{0\}$, then $B_1 \cdot (\operatorname{rad} A) = \{0\}$, and if $\bar{B}_1 \cdot (\operatorname{rad} \bar{A}) \cdot \bar{B}_i = \{0\}$ for* *all* $2 \leq i \leq t$*, then* $B_1 \cdot (\mathrm{rad}\, A) \cdot B_i = \{0\}$ *for all* $2 \leq i \leq t$*. Symmetrically, if* $(\mathrm{rad}\, \bar{A}) \cdot \bar{B}_1 = \{0\}$*, then* $(\mathrm{rad}\, A) \cdot B_1 = \{0\}$*, and if* $\bar{B}_i \cdot (\mathrm{rad}\, \bar{A}) \cdot \bar{B}_1 = \{0\}$ *for all* $2 \leq i \leq t$*, then* $B_i \cdot (\mathrm{rad}\, A) \cdot B_1 = \{0\}$ *for all* $2 \leq i \leq t$*.*

Proof. The fact that $\bar{A}$ has either minimal rank or minimal rank plus one follows from Corollary 1. But since $B = A/\mathrm{rad}\, A$ has minimal rank plus one, $\bar{A}$ cannot have minimal rank, because otherwise B would have minimal rank, too, by Lemma 4.

The second claim—$\bar{B}_1 \cdot (\mathrm{rad}\, \bar{A}) = \{0\}$ implies $B_1 \cdot (\mathrm{rad}\, A) = \{0\}$ and $\bar{B}_i \cdot (\mathrm{rad}\, \bar{A}) \cdot \bar{B}_1 = \{0\}$ for all $2 \leq i \leq t$ implies $B_i \cdot (\mathrm{rad}\, A) \cdot B_1 = \{0\}$ for all $2 \leq i \leq t$—is independent of the complexity of A. For the proof, we therefore refer to [6]. □

The next lemma shows that under the assumption that $B_1 \cdot (\mathrm{rad}\, A) \neq \{0\}$, then we can construct certain algebras of minimal rank or minimal rank plus one. (The case $(\mathrm{rad}\, A) \cdot B_1 \neq \{0\}$ is treated symmetrically.) We then show that the constructed algebras cannot have minimal rank or minimal rank plus one, if $B_1 = \mathbb{H}$.

Let E and F be algebras and let M be an E-left module and an F-right module. We call M a (E, F)-bimodule, if $(em)f = e(mf)$ for all $e \in E$, $m \in M$, and $f \in F$. If $E = F$, then we call M an E-bimodule.

If $E \cong D^{n \times n}$ is a simple algebra, then every E-left module M is isomorphic to $D^{n \times i}$ for some i. For right modules, the symmetric statement holds.

Lemma 13. *Let* A *be an algebra of minimal rank plus one,* B *a subalgebra of* A *with* $A = B \oplus \mathrm{rad}\, A$ *and* $B \cong A/\mathrm{rad}\, A$*, and* $B = B_1 \oplus \cdots \oplus B_t$ *with simple algebras* B_τ *and* $B_1 = \mathbb{H}$*. Assume that* $B_1 \cdot (\mathrm{rad}\, A) \neq \{0\}$*. Then either* $B_1(\mathrm{rad}\, A)B_i = \{0\}$ *for all* $2 \leq i \leq t$ *or there is a* q *with* $2 \leq q \leq t$ *and a nonzero* (B_1, B_q)*-bimodule* M *such that the algebra* $B_1 \times B_q \times M$ *(as vector spaces) with multiplication law* $(a, b, x) \cdot (a', b', x') = (aa', bb', ax' + xb')$ *has minimal rank plus one.*

Proof. By Lemma 12, we may assume w.l.o.g. that $(\mathrm{rad}\, A)^2 = \{0\}$. We now decompose $\mathrm{rad}\, A$ into twosided ideals as

$$\mathrm{rad}\, A = \bigoplus_{1 \leq \sigma, \tau \leq t} B_\sigma (\mathrm{rad}\, A) B_\tau.$$

If $B_1(\mathrm{rad}\, A)B_i = \{0\}$ for all $2 \leq i \leq t$, then we are done. Otherwise, there is some $q \geq 2$ such that $B_1 \cdot (\mathrm{rad}\, A) \cdot B_q \neq \{0\}$. W.l.o.g. $q = 2$. Let

$$I = \bigoplus_{(\sigma, \tau) \neq (1,2)} B_\sigma (\mathrm{rad}\, A) B_\tau.$$

It is easy to verify that I is a twosided ideal of A. Since A is of minimal rank plus one and $I \subseteq \mathrm{rad}\, A$, A/I is of minimal rank or minimal rank plus one by Lemma 4, too. We have

$$A/I \cong B_1 + \cdots + B_t + B_1(\mathrm{rad}\, A)B_2 \qquad \text{and} \qquad \mathrm{rad}(A/I) \cong B_1(\mathrm{rad}\, A)B_2.$$

By Lemma 12,

$$A/I \cong (B_1 + B_2 + B_1(\operatorname{rad} A)B_2) \times B_3 \times \cdots \times B_t.$$

To see this, note that if E and F are subalgebras of some algebra such that $E \cdot F = F \cdot E = \{0\}$, then $E + F \cong E \times F$. By Corollary 1, the algebra A/I is of minimal rank plus one only if $B_1 + B_2 + B_1(\operatorname{rad} A)B_2$ is of minimal rank or minimal rank plus one. Note that $B_1(\operatorname{rad} A)B_2$ is a nonzero (B_1, B_2)-bimodule. Moreover, it is easily checked that the algebra $B_1 + B_2 + B_1(\operatorname{rad} A)B_2$ obeys the multiplications law stated in the assertion of the lemma. □

7.2. Lower bounds

Next, we will show that the algebra $B_1 + B_2 + B_1(\operatorname{rad} A)B_2$, constructed in the previous lemma, is neither of minimal rank nor of minimal rank plus one.

Lemma 14. *Let D be some $\mathbb{R}$-division algebra and $M \neq \{0\}$ be a $(\mathbb{H}, D)$-bimodule. Let $A = \mathbb{H} \times D \times M$ be equipped with the multiplication $(a, b, m)(a', b', m') = (aa', bb', am' + mb')$. Then*

$$R(A) \geq 2 \dim A + \frac{1}{2} \dim M - 2.$$

In particular, A is neither of minimal rank nor of minimal rank plus one.

Proof. Let $\beta = (f_1, g_1, w_1; \ldots; f_r, g_r, w_r)$ be an optimal bilinear computation for A. Let $n = \dim A$ and $d = \dim D$. We can assume that $\operatorname{lin}\{w_1, \ldots, w_d\} \oplus \mathbb{H} \times \{0\} \times M = A$. Let $Y = \operatorname{lin}\{w_1, \ldots, w_{d-1}\}$. Obviously, β separates $(\{0\}, \{0\}, Y)$. Furthermore, β also separates $(\{0\} \times D \times M, \{0\}, Y)$. Otherwise, there is a nonzero $a = (0, a_1, a_2) \in \{0\} \times D \times M$, such that $a \cdot A \subseteq Y$. If $a_1 \neq 0$, then $a \cdot A$ has dimension at least d, a contradiction. If $a_1 = 0$ and $a_2 \neq 0$, then $a \cdot A \subseteq \{0\} \times \{0\} \times M$. This means $a \cdot A \cap Y = \{0\}$, a contradiction.

Let ϕ be the restriction of the multiplication in A to $A \times (\mathbb{H} \times \{0\} \times M)$. Let β' be the corresponding restriction of β. β' is obviously a computation for ϕ. Since this restriction only affects the g_ν, β' still separates $(\{0\} \times D \times M, \{0\}, Y)$. Let π be a projection along Y onto $\mathbb{H} \times \{0\} \times M$. By Lemma 1, we have

$$R(A) \geq R(\pi \circ \phi) + n - 4 + d - 1.$$

But $\pi \circ \phi$ is nothing else than the multiplication map of the $\mathbb{H}$-left module $\mathbb{H} \times M$. A slight extension of Lemma 5 (cf. [3, Remark after Lemma 8]) shows that

$$R(\phi) \geq 1 + \frac{3}{2}(4 + \dim M).$$

Altogether, we get

$$R(A) \geq n - 4 + d + \frac{3}{2}(4 + \dim M) = 2n + \frac{1}{2} \dim M - 2 > 2n - 1,$$

since $\dim M \geq 4$. □

The previous lemma settles the case that $B_2 = \mathbb{R}$ or $B_2 = \mathbb{C}$. It remains the case $B_2 = \mathbb{R}^{2\times 2}$. It is treated by the next lemma.

Lemma 15. *Let $A = \mathbb{H}\times\mathbb{R}^{2\times 2}\times M$ be equipped with the multiplication $(a, b, m)(a', b', m') = (aa', bb', am' + mb')$. Then A is neither of minimal rank nor of minimal rank plus one.*

Proof. M must be isomorpic to some $\mathbb{H}^{1\times i}$ as a left module and to some $\mathbb{R}^{j\times 2}$ as a right module. The smallest such module is $\mathbb{H}^{1\times 2}$, so $\dim M \geq 8$.

We have $R_{\mathbb{R}}(A) \geq R_{\mathbb{C}}(A\otimes\mathbb{C})$. $A\otimes C$ is isomorphic to $\mathbb{C}^{2\times 2}\times\mathbb{C}^{2\times 2}\times(M\otimes\mathbb{C})$ with the multiplication as given above. From [4, Lemma 8.8], we get

$$R_{\mathbb{C}}(A \otimes \mathbb{C}) \geq 2 \dim A + \frac{1}{4} \dim_{\mathbb{C}}(M \otimes \mathbb{C}) - 2.$$

The right hand side is $> 2 \dim A - 1$, since $\dim_{\mathbb{C}}(M \otimes \mathbb{C}) \geq 8$. □

7.3. Proof of Theorem 5

By Lemmas 11 and 13 and the two lower bounds in the previous section, it follows that

$$A \cong (B_1 + B_1(\operatorname{rad} A)B_1) \times (B' + B'(\operatorname{rad} A)B'),$$

where $B_1 = \mathbb{H}$. The next lemma shows, that $B_1(\operatorname{rad} A)B_1 = \{0\}$.

Lemma 16. *Let $M \neq \{0\}$ be a $\mathbb{H}$-bimodul and let $A = \mathbb{H} \times M$ be the algebra equipped with the multiplication $(a, m)(a', m') = (aa', am' + ma')$. Then*

$$R(A) > \frac{5}{2} \dim A - 3.$$

In particular, A is neither of minimal rank nor of minimal rank plus one.

Proof. Let $\beta = (f_1, g_1, w_1; \ldots; f_r, g_r, w_r)$ be a bilinear computation for A. Assume that $f_1, \ldots, f_n$ is a basis and let $x_1, \ldots, x_n$ be its dual basis. By [4, Lemma 7.2], there are indices ν_1, ν_2, ν_3 such that after sandwiching, $x_{\nu_1} = 1$ and $[x_{\nu_2}, x_{\nu_3}]$ is invertible. Now Lemma 5 implies

$$r \geq n - 3 + n + \frac{1}{2}n = 2n + \frac{1}{2}n - 3 > 2n,$$

since $n = \dim A = 4 + \dim M \geq 8$. □

Now we have $A = \mathbb{H} \times A'$ where $A' = B' + B'(\operatorname{rad} A)B'$. It just remains to show that A' has minimal rank.[3] But this follows from (1), since (1) yields

$$2 \dim \mathbb{H}' + 2 \dim A' - (t_{A'} + 1) + 1 = R(\mathbb{H} \times A') \geq 2 \dim \mathbb{H} + R(A'),$$

where $t_{A'}$ is the number of twosided ideals of A'. Note that $t_{A'}+1$ is the number of twosided ideals of A. This completes the proof of Theorem 5.

[3]It could be the case that $\mathbb{H}$ and A' do not fulfill the addivity conjecture, that is, $\mathbb{H}$ and A' and $\mathbb{H} \times A'$ could be of minimal rank plus one.

8. *B* has minimal rank

If B has minimal rank, then we have the following partial result. We call an $\mathbb{R}$-algebra C *superbasic* if $C/\operatorname{rad} C \cong \mathbb{R} \times \cdots \times \mathbb{R}$.

Since we only have a partial result and the techniques are similar to the ones in Section 7, we only sketch the proof and point out the differences. We first decompose A as in Section 7.1. It is easy to show an analogue of Lemma 13. Now $B_1 = \mathbb{R}^{2\times 2}$. We get algebras $C = \mathbb{R}^{2\times 2} \times B_q \times M$ with $B_q \in \{\mathbb{R}, \mathbb{C}, \mathbb{R}^{2\times 2}\}$ and have to show that none of these algebras has minimal rank plus one:

$B_q = \mathbb{R}$: [4, Lemma 8.7] gives a lower bound $R(C) \geq 2\dim C - 4 + m$, where $m = 2 + \frac{1}{2}\dim M$. If $\dim M \geq 4$, then C does not have minimal rank plus one. Since $\dim M$ is divisible by two, the case $\dim M = 2$, that is, $m = 3$ remains. The proof of [4, Lemma 8.7] uses the lower bound of 3m + 1 for the multiplication of 2×2 by $2 \times m$-matrices. For the case $m = 3$, Alekseyev [2] shows an improved lower bound of 11. Thus C does not have minimal rank plus in this case, too.

$B_q = \mathbb{C}$: Here, we again go over to the splitting field: We have $R_{\mathbb{R}}(C) \geq R_{\mathbb{C}}(C \otimes \mathbb{C})$. But $C \otimes \mathbb{C} = \mathbb{C}^{2\times 2} \times \mathbb{C} \times \mathbb{C} \times (M \otimes \mathbb{C})$. We get rid of one of the $\mathbb{C}$'s by using Lemmas 4 and 3. For the remaining algebra, we can exploit [4, Lemma 8.7] as in the first case.

$B_q = \mathbb{R}^{2\times 2}$: [4, Lemma 8.8] gives a lower bound $R(C) \geq 2\dim C - 4 + m$, where $m = 2 + \frac{1}{4}\dim M$. If $\dim M \geq 8$, then C does not have minimal rank plus one. Since $\dim M$ is divisible by four, the case $\dim M = 4$ remains. Again, the proof of [4, Lemma 8.7] uses the lower bound of 3m + 1 for the multiplication of 2×2 by $2 \times m$-matrices. For $m = 3$, i.e., $\dim M = 4$, we can again use the improved lower bound of 11.

Now we are in the situation that

$$A \cong (B_1 + B_1(\operatorname{rad} A)B_1) \times \cdots \times (B_i + B_i(\operatorname{rad} A)B_i) \times (B' + B'(\operatorname{rad} A)B'),$$

where $B_1 = \cdots = B_i = \mathbb{R}^{2\times 2}$ and $B'/\operatorname{rad} B' = \mathbb{R}^{t-i}$, that is, B' is superbasic. We would like to show that $B_1(\operatorname{rad} A)B_1 = \cdots = B_i(\operatorname{rad} A)B_i = \{0\}$.

$C := B_1 + B_1(\operatorname{rad} A)B_1$ has the structure $\mathbb{R}^{2\times 2} \times M$ with the multiplication map $(a, m)(a', m') = (aa', am' + a'm)$. $\dim M$ is divisible by 4. If $\dim M \geq 8$, then [4, Lemma 8.6] shows that C is not of minimal rank plus one. If $\dim M = 4$, then $C = \mathbb{R}^{2\times 2} \times \mathbb{R}^{2\times 2}$ with the multiplication map as above. For this algebra, we know a lower bound of 16. The technique of [4, Lemma 8.6] can also be extended to products of such algebras yielding a lower bound of 32 for a product of two. This show that there is at most one such factor.

To eliminate it completely from the considerations, we have to show the lower bound of 17. This seems plausible, since C consists of the matrix product $\langle 2, 2, 4\rangle$, for which we know a lower bound of 13, plus four additional bilinear forms.

If we could eliminate the case above, too, we would observe that the superbasic algebra $B' + B'(\operatorname{rad} A)B'$ has to have minimal rank plus one, because otherwise, A would not be of minimal rank plus one.

Currently, we do not know how to prove the lower bound of 17 nor do we know the structure of the superbasic algebras of minimal rank plus one.

9. Conclusions

Two natural questions arise. First, can we extend our results to other fields than $\mathbb{R}$? And second, can we get a full characterization of the algebras of minimal rank plus one (with radical)?

Over $\mathbb{R}$, there are only two nontrivial division algebras, $\mathbb{C}$ and $\mathbb{H}$. We used this fact several times in our proofs. Over $\mathbb{Q}$, there are more division algebras. The key question to solve the problem over $\mathbb{Q}$ is the following. For any numbers a, b, we can define quaternion algebras $H(a, b)$. Over $\mathbb{R}$, they are all either isomorphic to $\mathbb{R}^{2\times 2}$ or $\mathbb{H}$. Over $\mathbb{Q}$, the situation is more complicated. Question: What is $R_{\mathbb{Q}}(H(a, b))$ (in dependence on a, b)? If $H(a, b)$ is a division algebra, then it is clear that its rank is ≥ 8, since it is not a division algebra of minimal rank. The questions is whether 8 bilinear products are also sufficient.

For the second question, we first have to determine the bilinear complexity of the algebra $\mathbb{R}^{2\times 2} \times \mathbb{R}^{2\times 2}$ with multiplication map $(a, m)(a', m') = (aa', am' + a'm)$. We conjecture that it is not of minimal rank plus one. Second, we have to characterize the superbasic algebras of minimal rank plus one. We conjecture that these algebras have a richer structure. Two examples for these algebras are the algebra of upper triangular 3×3-matrices, a noncommutative algebra, and the commutative algebra $\mathbb{R}[X, Y]/(X^3, X^2Y, XY^2, Y^3)$.

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