Complexity of the Bollobás-Riordan Polynomial

Exceptional Points and Uniform Reductions

Markus Bläser[a], Holger Dell[b,⋆], and Johann A. Makowsky[c,⋆⋆]

[a]Saarland University, Saarbrücken, Germany
mblaeser@cs.uni-sb.de
[b]Humboldt University of Berlin, Berlin, Germany
dell@informatik.hu-berlin.de
[c]Technion-Israel Institute of Technology, Haifa, Israel
janos@cs.technion.ac.il

Abstract. The coloured Tutte polynomial by Bollobás and Riordan is, as a generalization of the Tutte polynomial, the most general graph polynomial for coloured graphs that satisfies certain contraction-deletion identities. Jaeger, Vertigan, and Welsh showed that the classical Tutte polynomial is #**P**-hard to evaluate almost everywhere by establishing reductions along curves and lines.

We establish a similar result for the coloured Tutte polynomial on integral domains. To capture the algebraic flavour and the uniformity inherent in this type of result, we introduce a new kind of reductions, *uniform algebraic reductions*, that are well-suited to investigate the evaluation complexity of graph polynomials. Our main result identifies a small, algebraic set of exceptional points and says that the evaluation problem of the coloured Tutte is equivalent for all non-exceptional points, under polynomial-time uniform algebraic reductions.

1 Introduction

Graph polynomials map directed or undirected graphs to polynomials in one or more variables, such that this mapping is invariant under graph isomorphisms. Their purpose is to study the combinatorial properties of graphs using algebraic and analytic properties of the associated polynomials. Probably the most famous graph polynomials are the *chromatic polynomial* $\chi(G;x)$ and its generalization, the *Tutte polynomial* $T(G;x,y)$. The chromatic polynomial is the polynomial in the variable x that counts the number of proper x-colourings of a given undirected graph (cf. [8] for an extensive modern exposition). Surprisingly, $\chi(G;-1)$ has a combinatorial interpretation, too: It counts the number of acyclic orientations [17].

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Certain evaluations of the Tutte polynomial T in two variables x and y have interpretations in different fields of combinatorics. For example, $T(G;1,1)$ counts the number of spanning trees, $T(G;1,2)$ counts the number of spanning subgraphs of an undirected graph G. Similarly the number of nowhere-zero flows, acyclic and Eulerian orientations can be obtained. Furthermore, the chromatic polynomial and the Jones polynomial of an alternating link can be derived from the Tutte polynomial via suitable substitutions and very simple algebraic transformations [5, 18].

Due to its rich combinatorial content, it is natural to analyze variations and generalization of the Tutte polynomial. Bollobás and Riordan [6] introduce the *coloured Tutte polynomial* and prove that it is the *most general* graph invariant that satisfies certain contraction-deletion identities. Related are the polynomials by Kauffman [11] and Sokal [16]. While the classical Tutte polynomial is in two variables, the coloured Tutte polynomial is defined on edge-coloured graphs, introducing four variables for every colour, and also some additional variables for initial conditions.

The purpose of this paper is the complexity analysis of and the reducibilities between evaluations of the coloured Tutte polynomial. For this, we propose a new kind of reductions, *(uniform) algebraic reductions*, that seems to be more suited for the complexity analysis of graph polynomials: Graph polynomials have two components, a combinatorial one, the graph, and an algebraic one, the values. So far, only the usual polynomial-time many-one or Turing reductions have been used for the complexity analysis. If one wants to talk about evaluations at irrational points like $\sqrt{2}$ or some root of unity –points that have meaningful interpretations for certain graph polynomials– there is no natural way of representing these points in a discrete setting. Jaeger, Vertigan, and Welsh [10] just adjoin the value they are interested in to $\mathbb{Q}$, but also admit that this is an ad hoc solution. Our reductions take care of this issue by also having two parts.

The combinatorial part of our reductions transforms the graph using a usual polynomial-time computable function from $\Sigma^* \to \Sigma^*$, mapping encodings of graphs to encodings of graphs. The algebraic part transforms the evaluation points and values in polynomial time, but these transformations are now restricted to be *rational* mappings and can be naturally extended to $\mathbb{C}$.

Previous Results. Jaeger, Vertigan, and Welsh [10] have shown that, except along one hyperbola and at four special rational and five special complex points, computing the Tutte polynomial is #**P**-hard. To show this, they construct for each non-exceptional evaluation point (a,b) a reduction to a point where evaluating the Tutte polynomial is already known to be #**P**-hard. All their reductions are very similar and depend only, and in some sense uniformly, on the point (a,b). However, this uniformity is not spelled out in their paper.

Recently, Lotz and Makowsky [13] proved that the coloured Tutte polynomial is complete for Valiant's algebraic complexity class **VNP**, and Goldberg and Jerrum [9] showed that the classical Tutte polynomial is inapproximable for large parts of the Tutte plane. Although not spelling out the inherent algebraicity and uniformity, many reductions in graph polynomials are of that type, among them are some reductions in matching polynomials [1], in the interlace polynomial [3], and in the cover polynomial [2].

Our Contribution. In the first main part, we introduce the notion of uniform and non-uniform algebraic reductions which spell out what Jaeger, Vertigan, and Welsh [10] had in mind in capturing combinatorial and algebraic aspects of graph polynomials (Sec. 3). For these reduction types, we prove that "#**P**-complete" graph polynomials can be *uniformly* reduced to any #**P**-hard numerical graph invariant (Sec. 4).

In the second main part, we establish the #**P**-hardness of the coloured Tutte polynomial under *uniform algebraic* reductions on all but a few exceptional evaluation points (Sec. 7 to 9). We also show in Sec. 7 how to carry over the inapproximability results of Goldberg and Jerrum [9] to the coloured Tutte polynomial using a simple approximation-preserving reduction from the classical Tutte polynomial.

The situation at the exceptional points for the coloured Tutte polynomial is less clear than for the classical Tutte polynomial, because of the larger number of possible colours involved. It seems that evaluating the coloured Tutte polynomial at the exceptional points is computable in polynomial time, as is the case in the classical Tutte polynomial. However, in [10] this is proven with sometimes very different proofs, which do not exhibit a common feature.

Our results also confirm the Uniform Difficult Point Conjecture [15] for the coloured Tutte polynomial, and our reductions can be used to analyze graph polynomials in a more general context, as described in [14, 15].

2 Preliminaries

Coloured graphs. Let $\mathbb{N} = \{0, 1, \dots\}$. The graphs in this paper are undirected *multigraphs* $G = (V, E)$ with parallel edges and loops allowed. By Λ, we denote a fixed finite set of colours, and $c : E \to \Lambda$ is called *colouring*. We denote by $\mathcal{G}$ the set of all graphs G and by $\mathcal{G}_c$ the set of all coloured graphs (G, c). We write $n(G)$ for the number of vertices, $m(G)$ for the number of edges, and $k(G)$ for the number of connected components of G. Two coloured graphs are called *isomorphic* if there is a bijective mapping on the vertices that transforms one graph into the other, thereby maintaining the colours.

Polynomials. Polynomials $p(x_1, \dots, x_v)$ are elements of a polynomial ring $\mathbb{Q}[x_1, \dots, x_v]$. We write $\mathbb{Q}(x_1, \dots, x_v)$ for its field of fractions.

Any univariate polynomial can be interpolated if sufficiently many point-value pairs are known. For multivariate polynomials, this is not always true, since the points must also be positioned nicely, e.g. in a grid. If, for a bivariate polynomial $p(x, y)$ of maximal degree d, the values at the points (x_α, y_α) for $\alpha = 1, \dots, n$ with $n = (d + 1)^2$ are known, p can be interpolated if, in addition, the bivariate Vandermonde matrix V_2 is non-singular:

$$V_2 = \begin{bmatrix} 1 & x_1 & y_1 & x_1y_1 & \cdots & x_1^i y_1^j & \cdots & x_1^d y_1^d \\ 1 & x_2 & y_2 & x_2y_2 & \cdots & x_2^i y_2^j & \cdots & x_2^d y_2^d \\ \vdots & & & & & & & \vdots \\ 1 & x_n & y_n & x_ny_n & \cdots & x_n^i y_n^j & \cdots & x_n^d y_n^d \end{bmatrix}.$$

Graph invariants. A *graph invariant* is a function $f : \mathcal{G} \to F$, mapping elements from $\mathcal{G}$ to some set F, such that all pairs of isomorphic graphs G and G' have the same image under f. If $F \subseteq \mathbb{Q}$, then f is called a *numeric graph invariant*. A *parameterized numeric graph invariant (PNGI)* is a function $f : \mathcal{G} \times \mathbb{N}^v \to \mathbb{N}$ which is invariant under graph isomorphisms. If, for each $G \in \mathcal{G}$, the function $f(G; \_)$ is a polynomial, then f is called *graph polynomial*. In this case, f has a natural extension to $\mathbb{C}$, which we use sometimes. Graph invariants and PNGI's for coloured graphs are defined in an analogous manner.

The *chromatic polynomial* $\chi(G; x)$ is the polynomial in x with the property that $\chi(G; Q)$, for $Q \in \mathbb{N}$, is the number of ways to colour the vertices of a graph with Q colours such that adjacent vertices have different colours.

3 Uniform algebraic reductions for graph polynomials

We want to study reducibilities between evaluations of PNGI's, that is, between the parameter-free numeric graph invariants $f(\boldsymbol{k}) := f(\_; \boldsymbol{k})$ for fixed $\boldsymbol{k} \in \mathbb{C}^v$.

It is widely accepted that the Tutte polynomial is combinatorially speaking strictly more expressive than the chromatic polynomial. Under usual polynomial-time Turing reductions, however, both graph polynomial are equivalent since they are both #**P**-complete (if restricted to positive integers) and can thus be reduced to each other. We now introduce a notion of uniform reducibility which comes closer to capture the intuitively accepted hierarchy between graph polynomials and PNGI's.

Definition 1. *Let f and g be two PNGI's. Denote by v and w the numbers of variables of f and g, respectively. Let $\boldsymbol{x} = x_1, \ldots, x_v$ and $\boldsymbol{y} = y_1, y_2, \ldots$ be distinct variable symbols.*

(i) We say f algebraically reduces to g uniformly*, or $f \preccurlyeq^{\mathrm{P}}_{\mathrm{AU}} g$, if there exist*
  (a) a parameterized rational function $A : \mathcal{G} \to \mathbb{Q}(\boldsymbol{x}, \boldsymbol{y})$,
  (b) functions $r : \mathbb{N} \times \mathcal{G} \to \mathcal{G}$, and
  (c) parameterized rational substitutions *$\sigma : \mathbb{N} \to (\mathbb{Q}(\boldsymbol{x}))^w$,*
  all polynomial-time computable, such that, for every $G \in \mathcal{G}$, the following identity holds for all possible values of $\boldsymbol{x}$:

$$f(G; \boldsymbol{x}) = A(G)\Big[y_i := g\big(G_i; \boldsymbol{x}_i\big)\Big],$$

  where $G_i = r(i, G)$ and $\boldsymbol{x}_i = \sigma(i)$. The brackets indicate that the variables y_i of the preceding polynomial are substituted by $g(G_i; \boldsymbol{x}_i)$. So basically, for given G, we express $f(G; \boldsymbol{x})$ in terms of a rational expression in $\boldsymbol{x}$ and $y_i = g(G_i; \boldsymbol{x}_i)$, where the $\boldsymbol{x}_i$ are again rational in $\boldsymbol{x}$.

(ii) *If all the graph transductions* $r(i, \_)$ *are the identity we say, that* f *is a* substitution instance of g *and we write* $f \preccurlyeq^{\mathrm{P}}_{SUB} g$.

(iii) *We say* f (algebraically) parsimoniously many-one reduces *to* g *in polynomial time, or* $f \preccurlyeq^{\mathrm{P}} g$, *if* $A(G) = y_1$ *for all* $G \in \mathcal{G}$.

(iv) *We say that* f *(non-uniformly)* algebraically reduces *to* g *if, for all* fixed $\boldsymbol{k} \in \mathbb{Q}^v$, *the parameter-free graph invariant* $f(\boldsymbol{k}) = f(\_; \boldsymbol{k})$ *is uniformly reducible to* g, *that is,* $f(\boldsymbol{k}) \preccurlyeq^{\mathrm{P}}_{\mathrm{AU}} g$ *for all* $\boldsymbol{k}$. *So basically, every* $\boldsymbol{k}$ *has its own* $A_{\boldsymbol{k}}$, $r_{\boldsymbol{k}}$, *and* $\sigma_{\boldsymbol{k}}$.

(v) *A meaningful way of algebraically reducing* any *function* $h : \Sigma^* \to \mathbb{N}$ *to a parameter-free numeric graph invariant* g *is by mapping the input* $x \in \Sigma^*$ *to graphs* $G_i = r(i, x)$ *and then write* $h(x)$ *as a rational function* $A(x)$ *in the oracle queries* $y_i = g(G_i)$.

The function r transforms the given graph G into graphs G_i, the function σ transforms the given point $\boldsymbol{x}$ into new points $\boldsymbol{x}_i$. Since the function A runs in polynomial time, only a polynomial number of the variables y_i can be introduced, that is, only a polynomial number of oracle queries $g(G_i; \boldsymbol{x}_i)$ take place. The outputs of these queries are combined using a rational expression in the helper variables y_i, which get later substituted by the $g(G_i; \boldsymbol{x}_i)$.

Our reductions are *uniform* because they are independent of $\boldsymbol{x}$. Our reductions are *algebraic* because the input substitutions before calling oracle g and the processing of the oracle outputs are bound to be rational transformations.

In general, the numbers of variables v and w do not have to be equal. One particularly important case is $w = 0$. Uniform algebraic reductions are similar to straight-line programs with oracle g (cf. [13], e.g.).

It is clear what it means that the functions r are polynomial time computable, and, as long as we work over $\mathbb{Q}$, this is also clear for σ or A, using a binary representation of the inputs. For ease of presentation, we restrict ourselves to $\mathbb{Q}$, but we can easily extend all our results to fields like $\mathbb{R}$ or $\mathbb{C}$ since rational functions over $\mathbb{Q}$ naturally extend to $\mathbb{R}$ or $\mathbb{C}$. Over $\mathbb{R}$ or $\mathbb{C}$, we can use the BSS-model [4] or a uniform variant of Valiant's model (cf. [7], e.g.) to define polynomial-time computable rational functions. Since these are unit cost models, the reductions become more powerful. However, the reductions in this work have the nice feature that their restrictions to $\mathbb{Q}$ are polynomial-time computable in ordinary bit models.

4 Hardness vs. Uniform Reductions

One way to state results about the complexity of a graph polynomial is to give a dichotomy theorem as in [10]: the points are partitioned completely into #**P**-hard points and easy points. But #**P**-hardness alone does not tell us, whether or not we can reduce the evaluation at one point to the evaluation at another point. Furthermore, it is not clear whether hard points capture the whole graph polynomial in a uniform way.

Recall that graph polynomials are functions $f : \mathcal{G} \times \mathbb{N}^v \to \mathbb{N}$. Thus, encoding the input of f in binary, the statement $f \in$ #**P** makes sense. Furthermore, for typical f one often knows some particular point $\boldsymbol{k}_0$ such that the numeric graph invariant $f(\boldsymbol{k}_0) = f(\_;\boldsymbol{k}_0)$ is #**P**-hard (in the Tutte polynomial, this might a-priori be the number of three-colourings). For such f, we prove that uniform and non-uniform reducibility to a parameter-free numeric graph invariant and the hardness of that invariant are equivalent.

Theorem 1. *Let $f \in$ #**P** be a graph polynomial in v variables such that $f(\boldsymbol{k}_0)$ is #**P**-hard under polynomial-time algebraic reductions, for some $\boldsymbol{k}_0$. Let g be a parameter-free numeric graph invariant. The following statements are equivalent:*

(i) It holds $f \preccurlyeq^{\mathrm{P}}_{\mathrm{AU}} g$.
(ii) For every $\boldsymbol{k}$, it holds $f(\boldsymbol{k}) \preccurlyeq^{\mathrm{P}}_{\mathrm{A}} g$.
*(iii) The function g is #**P**-hard under polynomial-time algebraic reductions.*

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are trivial.

Now we prove (iii) $\Rightarrow$ (i). From the assumptions, we get $f \preccurlyeq^{\mathrm{P}}_{\mathrm{A}} g$, where we see f as a #**P**-function and the reduction in the sense of part (v) in Definition 1. Thus, there exist a function $r' : \mathbb{N} \times (\mathcal{G} \times \mathbb{N}^v) \to \mathcal{G}$, mapping some parameter from $\mathbb{N}$ and instances of f to instances of g in polynomial time, and a parameterized rational function $A' : \mathcal{G} \times \mathbb{N}^v \to \mathbb{Q}(y_1, y_2, \dots)$ such that

$$f(G;\boldsymbol{j}) = A'(G,\boldsymbol{j})\Big[y_i := g\big(G_{i,\boldsymbol{j}}\big)\Big],$$

with $G_{i,\boldsymbol{j}} := r'(i,(G,\boldsymbol{j}))$ holds for all $(G,\boldsymbol{j}) \in \mathcal{G} \times \mathbb{N}^v$. We assume that the reduction is such that, for all G, the non-zero variables y_i are distinct for different $\boldsymbol{j}$, so we can write $y_{i,\boldsymbol{j}}$ for the variable to be replaced by $g(G_{i,\boldsymbol{j}})$.

Let d_G be the maximal degree of the polynomial $f(G;\boldsymbol{x})$. Given $G \in \mathcal{G}$ as an input, we compute $f(G;\boldsymbol{x})$ in a uniform way from g as follows. First we express the values $f(G;\boldsymbol{j})$ in terms of algebraic expressions $A'(G,\boldsymbol{j})$ in the variables $y_{i,\boldsymbol{j}} = g(G_{i,\boldsymbol{j}})$ using the reduction from above, and we do that for all $\boldsymbol{j}$ in the grid $\{0,1,\dots,d_G\}^v$. Formally, the graph transductions $r : \mathbb{N} \times \mathcal{G} \to \mathcal{G}$ interpret the parameter as a pair $\langle i,\boldsymbol{j}\rangle \in \mathbb{N}$ encoded as a nonnegative integer: We define the graph transductions to be $r(\langle i,\boldsymbol{j}\rangle, G) := G_{i,\boldsymbol{j}} = r'(i,(G,\boldsymbol{j}))$.

Next we apply multivariate interpolation to the point-value pairs $(\boldsymbol{j}, f(G;\boldsymbol{j}))$ of the grid, that is, we choose $A(G)$ to be an interpolation polynomial. Such a polynomial can be derived from the v-variate Vandermonde matrix V_v with evaluation points $\boldsymbol{j}$ and from the solution vector $\boldsymbol{b}$ with entries $f(G;\boldsymbol{j})$. Using the entries of $V_v^{-1}\boldsymbol{b}$ as the coefficients of a polynomial in $\boldsymbol{x}$, we get an explicit representation of $f(G;\boldsymbol{x})$ in terms of rational expressions in $\boldsymbol{x}$ and g. □

For PNGI's f that are not graph polynomials, the proof above does not work. Furthermore, it is enough to assume that f is in the closure of #**P** under polynomial-time algebraic reductions. Any natural graph polynomial seems to have this weaker property.

If we look at the special case $g = f(\_;\boldsymbol{k}')$ for fixed $\boldsymbol{k}'$, we can answer the question raised in the beginning of this section. Statement (iii) is the kind of hardness result that is widely used for graph polynomials. At first glance, one might conjecture (i) to be a much stronger property. But our theorem says that this intuition is not true if we have #**P**-hardness under algebraic reductions.

5 The Bollobás-Riordan Polynomial

The coloured Tutte polynomial [6] or *Bollobás-Riordan polynomial* is the most general graph polynomial which can be defined by a spanning tree expansion or a contraction-deletion identity.

The Bollobás-Riordan polynomial of a coloured graph G is a polynomial in the variables γ_k, X_λ, Y_λ, x_λ, and y_λ for $k \in \mathbb{N}$ and $\lambda \in \Lambda$. For a graph G, a colouring $c : E \to \Lambda$, and a (bijective) ordering of the edges $\Phi : E \to \{1, \cdots, m\}$, we define the weight of an edge e of colour λ with respect to a spanning forest $F \subset E$ to be

$$w(G, c, \Phi, F, e) = \begin{cases} X_\lambda & \text{if } e \text{ is internally active,} \\ Y_\lambda & \text{if } e \text{ is externally active,} \\ x_\lambda & \text{if } e \text{ is internally inactive,} \\ y_\lambda & \text{if } e \text{ is externally inactive.} \end{cases}$$

Here we say that an edge $\{u, v\} = e \in F$ is internally active if it is the first edge of E (with respect to Φ) that touches the connected components of u and v in $F - e$, and internally inactive, otherwise. An edge $e \in E - F$ is said to be externally active if it is the first edge of the unique cycle in $F \cup e$, and externally inactive, otherwise. The Bollobás-Riordan polynomial is defined as

$$T_{\text{col}}(G, c, \Phi) = \gamma_{k(G)} \sum_{F \subset E(G)} \prod_{e \in E} w(G, c, \Phi, F, e), \tag{1}$$

where the sum is over all spanning forests of G. In order to remove the dependence on the order, computation must be modulo some ideal I'_0. For an arbitrary ideal $I \supset I'_0$, we define the quotient ring in which the coloured Tutte polynomial lives as $\mathcal{R} = \mathbb{Z}\left[\gamma_k, X_\lambda, Y_\lambda, x_\lambda, y_\lambda : k \in \mathbb{N}, \lambda \in \Lambda\right] \big/ I$. In the case that $\mathcal{R}$ is an integral domain, Bollobás and Riordan [6] prove that T_{col} on $\mathcal{R}$ is a graph invariant, i.e. independent from the order Φ if and only if, in the polynomial ring $\mathcal{R}$, it holds either

$$X_\lambda y_\mu - y_\lambda X_\mu = x_\lambda Y_\mu - Y_\lambda x_\mu = x_\lambda y_\mu - y_\lambda x_\mu \tag{2}$$

for all colours λ and μ, or $\gamma_k = 0$ for all k, or $X_\lambda = Y_\lambda = 0$ for all colours λ. (The ideal I'_0 establishes exactly this situation). Let us write $T_{\text{col}}(G, c) = T_{\text{col}}(G, c, \Phi)$ in $\mathcal{R}$ for an arbitrary ordering Φ since the ordering is now negligible. We abbreviate $T_{\text{col}}(G) = T_{\text{col}}(G, c)$.

In what follows, we assume that I is chosen in such a way that $\mathcal{R}$ is an integral domain, that is, if $pq = 0$ then $p = 0$ or $q = 0$ in $\mathcal{R}$, for all $p, q \in \mathcal{R}$. Furthermore, we assume that we are *not* in the second case, that is, we assume $\gamma_k \neq 0$ and either $X_\lambda \neq 0$ or $Y_\lambda \neq 0$ for some k and λ. In the second case we would have $T_{\text{col}}(G) = 0$ for all graphs G which is not very interesting.

The *Tutte polynomial* $T(G; x, y)$ of a graph G is the instance of the Bollobás-Riordan polynomial with $X_\lambda = x$, $Y_\lambda = y$, $x_\lambda = y_\lambda = \gamma_k = 1$ for all λ and k. It is known that $\chi(G; x) = (-1)^{k(G)} T(G; 0, 1 - x)$ holds. The standard #**P**-hardness proof [12] for the chromatic polynomial χ actually uses *algebraic* reductions:

Lemma 1. *The numerical graph invariants* $\chi(\_;0)$, $\chi(\_;1)$, $\chi(\_;2)$ *are polynomial-time computable, and all other* $\chi(\_;Q)$ *are* $\#\mathbf{P}$*-hard under polynomial-time* algebraic *reductions.*

6 Evaluations

An *evaluation point* $\sigma : \mathcal{R} \to \mathbb{Q}$ is an arbitrary ring homomorphism mapping the variables that may occur in the coloured Tutte polynomial to rationals, such that $\sigma(x_\lambda y_\mu \gamma_k) \neq 0$ for all λ, μ, and k. Note that this restriction excludes only computationally trivial cases that can be eliminated in polynomial time by applying the contraction-deletion identity [6] recursively.

For the values of the variables of colour λ, we write

$$(a,b,c,d) = \sigma|_\lambda := (\sigma(X_\lambda), \sigma(Y_\lambda), \sigma(x_\lambda), \sigma(y_\lambda)).$$

We also write $\sigma|_{\neq\lambda}$ for the tuple of all other values (including $\sigma(\gamma_k)$ for all k).

Let us define three important invariants in the fraction field of $\mathcal{R}$:

$$q_\lambda := (X_\lambda - x_\lambda)/y_\lambda \ , \quad r_\lambda := (Y_\lambda - y_\lambda)/x_\lambda \ , \text{ and } \quad Q_\lambda := q_\lambda r_\lambda \ .$$

For an evaluation point σ, we define $q_\sigma = \sigma(q_\lambda)$, $r_\sigma = \sigma(r_\lambda)$, and $Q_\sigma = \sigma(Q_\lambda)$ for an arbitrary colour λ. Rather surprisingly, the following holds.

Lemma 2. *The values* q_σ*,* r_σ*, and* Q_σ *are well-defined, i.e., independent from the choice of* λ*.*

Proof. We prove the claim only for q_σ. Using (2), we compute in $\mathcal{R}$: $y_\lambda y_\mu q_\lambda = y_\lambda y_\mu (X_\lambda - x_\lambda)/y_\lambda = X_\lambda y_\mu - x_\lambda y_\mu = y_\lambda X_\mu - y_\lambda x_\mu = y_\lambda y_\mu (X_\mu - x_\mu)/y_\mu = y_\lambda y_\mu q_\mu$. The fraction field of $\mathcal{R}$ is an integral domain, and the claim follows. □

This lemma says that, for evaluation points from $\mathcal{R} \to \mathbb{Q}$, not all value combinations are allowed for the variables. To make this structure more concrete, we define the sets

$$L_{q,r} := \Big\{ (a,b,c,d) \in \mathbb{Q}^4 : qd = a - c \ \text{ and } \ rc = b - d \Big\},$$
$$P_{q,r} := \Big\{ \sigma : \sigma|_\lambda \in L_{q,r} \text{ for all } \lambda \in \Lambda \Big\}.$$

From Lemma 2, we immediately get the following observation.

Lemma 3. *The set* $\bigcup_{q,r\in\mathbb{Q}} P_{q,r}$ *and the set of all evaluations* $\mathcal{R} \to \mathbb{Q}$ *are equal.*

We define the counting problem of evaluating the Bollobás-Riordan polynomial as

$$\sigma T_{\mathrm{col}} : \mathcal{G}_c \to \mathbb{Q} \quad \text{with} \quad G \mapsto \sigma T_{\mathrm{col}}(G) := \sigma\big(T_{\mathrm{col}}(G)\big).$$

The numerical graph invariant σT_{col} evaluates the Bollobás-Riordan polynomial of a given coloured graph G over $\mathcal{R}$ at the point σ.

Choosing an appropriate variable substitution in the coloured Tutte polynomial yields the classical Tutte polynomial: We define a ring homomorphism $\varphi : \mathcal{R} \to \mathbb{Q}[x,y]$ with $\varphi(\gamma_k) = 1$ and $\varphi|_\lambda = (x,y,1,1)$ for all k and λ. Bollobás and Riordan [6] prove that $\varphi T_{\mathrm{col}}(G) = T(G;x,y)$ holds, where $T(G;x,y)$ is the (classical) Tutte polynomial of G.

7 Simple Reductions

Our first simple reduction shows that we can ignore the choice of $\sigma(\gamma_k)$ for the rest of this paper.

Lemma 4. *Let σ and σ' be two evaluation points with $\sigma|_\lambda = \sigma'|_\lambda$ for all colours λ. It holds $\sigma' T_{col} \preccurlyeq^{\mathrm{P}}_{\mathrm{A}} \sigma T_{col}$.*

Proof. Using (1) or alternatively (3.12) from [6], we drag γ_k out of the coloured Tutte polynomial, $\sigma' \left(T_{\mathrm{col}}(G)/\gamma_{k(G)}\right) = \sigma \left(T_{\mathrm{col}}(G)/\gamma_{k(G)}\right)$. We get $\sigma' T_{\mathrm{col}}(G) = \left(\sigma'(\gamma_{k(G)})/\sigma(\gamma_{k(G)})\right) \cdot \sigma T_{\mathrm{col}}(G)$. □

Our second simple reduction relies on the homogeneity of variables. From (1) one can see that $T_{\mathrm{col}}(G)$ is homogeneous in the X, Y, x, y-variables, with a summed degree of m in these variables. Furthermore, the variables X and x always have together a summed degree of $n - k$, and Y and y have a summed degree of $m - n + k$, correspondingly. Thus, for every s from the fraction field of $\mathcal{R}$,

$$\eta_s T_{\mathrm{col}}(G) = s^{m-n+k} T_{\mathrm{col}}(G),$$

where η_s is the ring homomorphism with $\eta_s(Y) = s \cdot Y$, $\eta_s(y) = s \cdot y$, and that leaves everything else identical. This gives us the following many-one reduction.

Lemma 5. *For an arbitrary point σ, we have $(\sigma \circ \eta_s) T_{col} \preccurlyeq^{\mathrm{P}}_{\mathrm{A}} \sigma T_{col}$ for all $s \in \mathbb{Q}$.*

Note that $q_{\sigma\circ\eta_s} = q_\sigma / s$ and $r_{\sigma\circ\eta_s} = r_\sigma \cdot s$ holds, meaning that q_σ and r_σ are in general not invariant under η_s. However, this is true for $Q_\sigma = Q_{\sigma\circ\eta_s}$.

Using the homogeneity reduction, we can easily see the following reduction.

Proposition 1. *Let σ be an evaluation point with $\sigma_\lambda = (a, b, c, d)$ for some colour λ. Then $T(a/c, b/d) \preccurlyeq^{\mathrm{P}}_{\mathrm{A}} \sigma T_{col}$.*

Proof. Given input G, we compute $T(G; a/c, b/d)$ from $\sigma T_{\mathrm{col}}(G)$: We assign colour λ to every edge, and we use Lemma 5 and its analogue for the X, x-variables to get $T(G; a/c, b/d) = \sigma T_{\mathrm{col}}(G) / \left(c^{n-k} d^{m-n+k}\right)$. □

This reduction is an approximation-preserving many-one reduction. Therefore, the inapproximability results by Goldberg and Jerrum [9] transfer immediately to the coloured Tutte polynomial. Although it also gives hardness immediately, as well, we prove it independently in the following, because the proof gives some insights into the structure of the Bollobás-Riordan Polynomial.

8 Interpolation using Parallel and Series Reduction

We use the parallel and series identities from Theorems 7 and 9 in [6] to obtain algebraic reductions for the coloured Tutte polynomial.

When using r_λ or q_λ in the following, we actually always stay in the ring $\mathcal{R}$ since the denominators cancel out.

Lemma 6. *Let G be a coloured graph, let λ be a colour, and let $G^{\alpha\text{-}fat\text{-}\lambda}$ be the graph obtained from G by replacing each edge of colour λ by α parallel edges of the same colour λ.*

Then $f_{\alpha,\lambda}\big(T_{col}(G)\big) = T_{col}\big(G^{\alpha\text{-}fat\text{-}\lambda}\big)$ where $f_{\alpha,\lambda}$ is the unique ring homomorphism:

$$\begin{aligned} f_{\alpha,\lambda}(X_\lambda) &= r_\lambda^{-1}(Y_\lambda^\alpha - y_\lambda^\alpha) + q_\lambda y_\lambda^\alpha, & f_{\alpha,\lambda}(Y_\lambda) &= Y_\lambda^\alpha, \\ f_{\alpha,\lambda}(x_\lambda) &= r_\lambda^{-1}(Y_\lambda^\alpha - y_\lambda^\alpha), & f_{\alpha,\lambda}(y_\lambda) &= y_\lambda^\alpha, \quad \textit{and} \\ f_{\alpha,\lambda}|_{\neq\lambda} &= id|_{\neq\lambda}. \end{aligned}$$

Proof (sketch). The proof is by induction on α. The induction step uses Theorem 7 from [6] which basically talks about the case $\alpha = 2$. □

Similarly, using Theorem 9 from [6], one can prove an analoguous lemma for series reduction. Let $g_{\beta,\lambda}$ be the ring homomorphism that moves points using the β-stretching of a graph. The two lemma together establish parsimonious reductions in the Bollobás-Riordan polynomial.

Lemma 7. *Let $\alpha, \beta \in \mathbb{N}_{>0}$, $\lambda \in \Lambda$, and $\sigma : \mathcal{R} \to \mathbb{Q}$ be an evaluation point.*
It holds $\big(\sigma \circ g_{\beta,\lambda} \circ f_{\alpha,\lambda}\big)T_{col} \preccurlyeq^{\mathrm{p}} \sigma T_{col}$.

In the remainder of this section we interpolate the coloured Tutte polynomial from the given data points $(\sigma \circ g_{\beta,\lambda} \circ f_{\alpha,\lambda})$, for different α and β (but fixed λ).

The main problem of the interpolation process is that, in contrast to the situation in the classical Tutte polynomial, the parallel and series reductions do *not* move evaluation points along a relatively simple one-dimensional variety. Instead, all we can say is that the points produced by these reductions lie in the variety $L_{q,r}$, which has *two* dimension.

We call an evaluation σ *stuck* in λ if, for $\sigma|_\lambda = (a, b, c, d)$, we have $|a| \in \{0, |c|\}$ and $|b| \in \{0, |d|\}$. For every fixed λ and $c, d \neq 0$, there are exactly nine stuck points.

Theorem 2. *For $q, r \in \mathbb{Q}$, let $\sigma \in P_{q,r}$ be an evaluation point that is not stuck in λ. Then, for all $\sigma' \in P_{q,r}$ with $\sigma'|_{\neq\lambda} = \sigma|_{\neq\lambda}$, it holds $\sigma' T_{col} \preccurlyeq^{\mathrm{p}}_{\mathrm{A}} \sigma T_{col}$.*

Proof (sketch). The rather technical proof works by showing that the Vandermonde matrix V_2 is non-singular for the data points given by the stretching or fattening reductions. □

For $c = d = 1$, this theorem specializes to the line interpolation theorem for the classical Tutte polynomial [10]. In a suitable model of computation, the theorem also works for $\mathbb{R}$ and $\mathbb{C}$.

9 Complexity of the Bollobás-Riordan Polynomial

The dichotomy theorem from the classical paper [10] says that evaluating the Tutte polynomial over $\mathbb{R}$ is #**P**-hard, except for the *special* points $(1, 1)$, $(0, -1)$, $(-1, 0)$, $(-1, -1)$ and one hyperbola $(x - 1)(y - 1) = 1$ in the Tutte plane, where this is easy. For the Bollobás-Riordan polynomial, one expects a similar dichotomy. The hyperbola becomes the variety $Q_\sigma = 1$, and we extend the notion of special points in a natural way: We say that σ is *special* if, for all colours λ and for $(a, b, c, d) = \sigma|_\lambda$, it holds

$$(a, b) \in \{(c, d), (0, -d), (-c, 0), (-c, -d)\}\,.$$

We classify the complexity of evaluating the Bollobás-Riordan polynomial in the following way.

Main Theorem. *Let σ be an evaluation point. It holds:*

*(i) If $Q_\sigma \neq 1$ and σ is not special, σT_{col} is #**P**-hard under Turing reductions.*
*(ii) If $Q_\sigma \neq 1$ and σ is not special, σT_{col} is #**P**-hard under algebraic reductions.*
(iii) If $Q_\sigma = 1$, then σT_{col} is polynomial-time computable.

Proof. (i) Let λ be a colour in which σ is non-special and write $(a, b, c, d) = \sigma|_\lambda$. By Proposition 1, $T(a/c, b/d) \preccurlyeq^{\mathrm{p}}_{\mathrm{A}} \sigma T_{\mathrm{col}}$. Since $(a/c, b/d)$ neither lies on the hyperbola $(x - 1)(y - 1) = 1$ nor is one of the special points $(1, 1)$, $(0, -1)$, $(-1, 0)$, $(-1, -1)$ in the Tutte plane, it is #**P**-hard to evaluate under Turing reductions [10].

(ii) There exists a colour λ in which σ is not stuck. This is because, for $Q_\sigma \notin \{0, 1, 2\}$, σ can only be stuck if it is of the form $(-c, -d, c, d)$ and therefore special. Let φ be the evaluation point with $\varphi|_\lambda = (1 - Q_\sigma, 0, 1, 1)$ and $\varphi|_{\neq\lambda} = \sigma|_{\neq\lambda}$. Both σ and φ are evaluation points in $P_{q,r}$, so we can apply Theorem 2 to get $\varphi T_{\mathrm{col}} \preccurlyeq^{\mathrm{p}}_{\mathrm{A}} \sigma T_{\mathrm{col}}$. The claim follows for $Q_\sigma \notin \{0, 1, 2\}$ because $\chi(Q_\sigma) = \varphi T_{\mathrm{col}}$ holds (cf. Sec. 5) and $\chi(Q_\sigma)$ is #**P**-hard by Lemma 1. For $Q_\sigma = 0, 2$, we can use the reduction from (i): Hardness of such points is proven in [10] by an *algebraic* reduction from reliability computations or from the permanent, both of which are problems that can be proven to be hard under *algebraic* reductions.

(iii) From [10], we know that $\varphi T_{\mathrm{col}} = T(2, 0)$ is polynomial-time computable, where $\varphi|_\mu := (2, 0, 1, 1)$ for all μ. Furthermore, $\varphi|_\mu$ is not stuck in any colour, which allows us to apply Theorem 2 in series to each colour to get the reduction $(\sigma \circ \eta_s) T_{\mathrm{col}} \preccurlyeq^{\mathrm{p}}_{\mathrm{A}} \varphi T_{\mathrm{col}}$. Here we choose $s := q_\varphi/q_\sigma = r_\sigma/r_\varphi$ in such a way, that $(\sigma \circ \eta_s) \in P_{q_\varphi, r_\varphi}$. To finish the proof, we apply Lemma 5 and obtain $\sigma T_{\mathrm{col}} = (\sigma \circ \eta_s \circ \eta_{1/s}) T_{\mathrm{col}} \preccurlyeq^{\mathrm{p}}_{\mathrm{A}} (\sigma \circ \eta_s) T_{\mathrm{col}}$. □

For the proof of (ii), we could have used (i) also for the cases where $Q_\sigma \neq 0, 2$ since the reductions in [10] are actually algebraic. Instead, we decided to give an independent proof.

By Theorem 1, any non-special point with $Q_\sigma \neq 1$ is as hard as the whole graph polynomial, even under polynomial-time uniform and algebraic reductions.

Corollary 1. *For any non-special evaluation point σ with $Q_\sigma \neq 1$, we have $T_{col} \preccurlyeq^{\mathrm{p}}_{\mathrm{AU}} \sigma T_{col}$. In particular, we have $\sigma' T_{col} \preccurlyeq^{\mathrm{p}}_{\mathrm{A}} \sigma T_{col}$ for all points σ'.*

Open Problem. *Is σT_{col} polynomial-time computable if σ is special?*

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