Assignment 10, Complexity Theory, WiSe 16/17

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The goal of this exercise is to prove Ladner's theorem, which shows the existence of an infinite set of languages that are in NP, but neither NP complete nor in P. Let $M_1, M_2, \ldots, M_i, \ldots$ be an enumeration of deterministic TMs that accept languages such that M_i is n^i time bounded on inputs of length n. Let $A \in \mathsf{NP}$, and define

$$B = \{x01^{g(|x|)-|x|-1} \mid x \in A\}$$

where g is a function defined inductively as follows:

Input: $n \in \mathbb{N}$
Output: $g(n)$

a) Set $i = 1$.

b) For all y with $|y| \leq \log\log n$ do

   (i) if $y \in L(M_i)$ and $y \notin B$ or vice versa, then $i := i + 1$.

c) return $g(n) = n^i$.

Exercise 10.1 Show that $g(n)$ can be computed time polynomial in n and $g(n)$, where n is input in unary.

Exercise 10.2 Prove that $B \leq_P A$.

Exercise 10.3 Show that, if $A \notin \mathsf{P}$, then $B \notin \mathsf{P}$.

Exercise 10.4 Show that if $A \notin \mathsf{P}$, then A is not polynomial time many-one reducible to B.

The above exercises prove Ladner's Theorem:

Theorem 1 (Ladner) Suppose $\mathsf{P} \neq \mathsf{NP}$. Let $A \in \mathsf{NP} \setminus \mathsf{P}$. Then there is a $B \in \mathsf{NP} \setminus \mathsf{P}$ such that $B \leq_P A$ and $A \not\leq_P B$.