Assignment 3, Complexity Theory, WiSe 16/17

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In this home work, we will give an alternative proof that `PARITY` does not have polynomial size and constant depth unbounded fanin circuits.

Let $X = \{x_1, \dots, x_n\}$ be a set of variables. A partial assignment α is a mapping $X \to X \cup \{0, 1\}$ with the property that for all $x \in X$, $\alpha(x) \in \{x, 0, 1\}$, i.e., some variables are substituted by some value, the other ones are left unchanged.

If a substitution assigns 0 to x_i and x_i is the input of some $\wedge$-gate, then we can remove this gate and replace it by the constant 0. The same is true for 1 and $\vee$-gates.

The Parity function on n inputs has the nice property that after a partial assignment, we still compute the Parity function or its negation on a smaller number of inputs.

Throughout this exercise, we will consider random partial assignments that independently set each variable x_i to 0 or 1, each with probability $(1 - 1/\sqrt{n})/2$, and leaves x_i unchanged with probability $1/\sqrt{n}$.

It is sufficient to solve 4 of the exercises.

Exercise 3.1 Show that

$$\Pr_{\alpha}[< \sqrt{n}/2 \text{ variables are unassigned}] \leq (\frac{2}{e})^{\sqrt{n}/2}.$$

(Hint: Chernoff bound)

Exercise 3.2 Let $Y \subseteq X$ with $c \leq |Y| \leq o(n^{1/c})$. Show that

$$\Pr_{\alpha}[\geq c \text{ variables of } Y \text{ are unassigned}] \leq n^{1-c/2}.$$

Exercise 3.3 Let $Y_1, \dots, Y_\ell \subseteq X$ be pairwise disjoint with $|Y_i| < \eta$ for all i. For large enough n, the probability that there is no i such that a random partial assignment assigns 0 to all variables in Y_i is $\leq (1 - 2^{-\eta})^\ell$.

Exercise 3.4 (*hard and lengthy!*) We say that a circuit has level-1-fanin bounded by r if all gates that have only variables as inputs (and not the output of any other gate) have size at most r. Show the following (by using random restrictions and removing trivialized gates):

a) If `PARITY` has polynomial size circuits of depth d, then `PARITY` has polynomial size circuits of depth d with level-1-fanin bounded by some constant r.
(Formally, we do not remove trivialized gates here but replace it by a gate with only one (constant) input.)

b) If `PARITY` has polynomial size circuits of depth $d \geq 3$ with level-1-fanin bounded by some constant $r \geq 2$, then `PARITY` has polynomial size circuits of depth d with level-1-fanin bounded by $r - 1$.

c) If `PARITY` has polynomial size circuits of depth $d \geq 1$, with level-1-fanin bounded by 1, then `PARITY` has polynomial size circuits of depth $d - 1$.

Exercise 3.5 Show that there is no family of depth 2 circuits with level-1-fanin bounded by $n-1$ that computes `PARITY`.

Exercise 3.6 Show that `PARITY` is not computed by polynomial size and constant depth unbounded fanin circuits.