Assignment 11, Complexity Theory, WS 13/14

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due: never

Exercise 11.1 A *clique* in an undirected graph $G = (V, E)$ is a subset $C \subseteq V$ such that $\{u, v\} \in E$ for all $u, v \in C$, $u \neq v$. A clique C is a k-clique if $|C| = k$. Let $\texttt{Clique} = \{(G, k) \mid G \text{ contains a } k\text{-clique}\}$.

a) Assign to each node v of $G = (V, E)$ a weight $w_v \in \{1, \dots, 2n\}$ uniformly at random. For a subset $U \subseteq V$, the weight of U is $w(U) = \sum_{u \in U} w_u$. Let $1 \leq k \leq |V|$. Show that with probability $\geq 1/2$, there is exactly one k-clique of *maximum* weight, if G has at least one k-clique.

b) Let $n = |V|$ and $1 \leq k \leq n$. Next replace each node $v \in V$ by a $(2nk + w_v)$-clique. For every edge $\{u, v\} \in E$, connect every node of the clique belonging to u with every node of the clique belonging to v. Let $G' = (V', E')$ be the resulting graph. Show that with probability $\geq 1/2$, there is an $1 \leq r \leq 2nk$ such that

$$G \text{ has no } k\text{-clique} \Longrightarrow G' \text{ has no } (2nk^2 + r)\text{-clique}.$$
$$G \text{ has a } k\text{-clique but no } (k+1)\text{-clique} \Longrightarrow G' \text{ has } \textit{exactly} \text{ one } (2nk^2 + r)\text{-clique}.$$

c) Show that if there is a polynomial time bounded deterministic Turing machine that on input (H, k) outputs a k-clique of H if H has exactly one k-clique (and otherwise can do what it wants), then $\mathsf{NP} = \mathsf{RP}$.

d) We know that $\#\texttt{SAT}$ is $\#\mathsf{P}$-complete under parsimonious reductions. Use this fact to deduce the Valiant–Vazirani theorem from c).

Exercise 11.2 Which of the following statements are true, which are false, which imply $\mathsf{P} = \mathsf{NP}$. Give short proofs of your answers.

a) There is an oracle A such that $\mathsf{NP}^A = \mathsf{PSPACE}^A$.

b) For all functions $f : \{0,1\}^n \to \{0,1\}$: If there is a multilinear polynomial p of degree $\geq \sqrt{n}$ such that $\Pr_{x \in \{0,1\}^n}[p(x) = f(x)] \geq 0.9$, then there is no unbounded fanin Boolean circuit of constant depth and polynomial size that computes f.

c) For all classes C: If C is closed under polynomial time Turing reductions, then $\exists\, \mathsf{BP}\text{-}\,\mathsf{C} \subseteq \mathsf{BP}\text{-}\,\exists \mathsf{C}$.

d) For all NP-relations R: If $\#R$ is $\#\mathsf{P}$-complete under polynomial time Turing reductions, then $L(R)$ is NP-complete under polynomial time Turing reductions.

e) If $\mathsf{P} = \mathsf{PSPACE}$, then $\mathsf{RP} = \mathsf{BPP}$.

Exercise 11.3 Prove that $\mathsf{PSPACE} \subseteq \mathsf{P}/\mathsf{poly}$ implies $\mathsf{PSPACE} = \mathsf{MA}$.
(Hint: The prover in the protocol of the proof that $\mathsf{IP} = \mathsf{PSPACE}$ is polynomomial space bounded.)