In the exercises, we study a new bootstrap problem under another formulation of the noise: the definition of the noise level at a white/blue node remains the same as in the lecture, while the noise level at a red node is the sum of the noise levels among all input gates that are not marked, that is:

\[
\ell(v) = \begin{cases} 
0 & \text{if } v \text{ is white,} \\
\max_{(u,v) \in E} \ell(u) \cdot 1_{V \setminus S}(u) & \text{if } v \text{ is blue,} \\
\sum_{(u,v) \in E} \ell(u) \cdot 1_{V \setminus S}(u) & \text{if } v \text{ is red.}
\end{cases}
\]

**Exercise 10.1 (10 Points)**

As a counterpart of an interesting path defined in the lecture, we now define an interesting tree, which is a connected subgraph of \( G \) with a tree structure, such that it contains exactly \( L + 1 \) red vertices.

Similar to Fact 2.1 in the lecture, show that a set of marked vertices is a feasible solution to the new bootstrap problem if and only if every interesting tree has a non-root vertex that is marked.

**Exercise 10.2 (10 Points)**

Provide a linear programming formulation for the new bootstrap problem based on Exercise 10.1.

**Exercise 10.3 (10 Points)**

Show that for the new bootstrap problem, the hardness result still holds:

Let \( L \geq 2 \) be an integer parameter. For any \( \epsilon > 0 \), it is NP-hard to approximate the new bootstrap problem within a factor of \( L - \epsilon \), assuming the Unique Games Conjecture.

**Exercise 10.4 (10 Points)**

Can you round the linear program in Exercise 10.2 to obtain an \( L \)-approximation for the new bootstrap problem? Either explain the difficulties in extending the rounding argument from the lecture, or if you succeed in obtaining an \( L \)-approximation algorithm for the new bootstrap problem, then congratulations: You can then publish a research paper!