Saarland Informatics Campus

Assignment 7, Approximation Algorithms
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http://www-cc.cs.uni-saarland.de/course/61/

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In the exercises, we study the *uniform labeling problem*: we are given a graph $G = (V, E)$, costs $c_e \geq 0$ for all $e \in E$, and a set of labels L that can be assigned to the vertices of V. There is a nonnegative cost $c_v^i \geq 0$ for assigning label $i \in L$ to vertex $v \in V$, and an edge $e = (u, v)$ incurs cost c_e if u and v are assigned different labels. The goal of the problem is to assign each vertex in V a label so as to minimize the total cost.

We give an integer programming formulation of the problem. Let the variable $x_v^i \in \{0, 1\}$ be 1 if vertex v is assigned label $i \in L$, and 0 otherwise. Let the variable z_e^i be 1 if exactly one of the two endpoints of the edge e is assigned label i, and 0 otherwise. Then the integer programming formulation is as follows:

$$
\begin{aligned}
\text{minimize } & \frac{1}{2}\sum_{e\in E} c_e \sum_{i\in L} z_e^i + \sum_{v\in V, i\in L} c_v^i x_v^i \\
\text{subject to } & \sum_{i\in L} x_v^i = 1, && \forall v \in V, \\
& z_e^i \geq x_u^i - x_v^i, && \forall e = (u, v) \in E, \forall i \in L, \\
& z_e^i \geq x_v^i - x_u^i, && \forall e = (u, v) \in E, \forall i \in L, \\
& z_e^i \in \{0, 1\}, && \forall e \in E, \forall i \in L, \\
& x_v^i \in \{0, 1\}, && \forall v \in V, \forall i \in L.
\end{aligned}
$$

Exercise 7.1 (8 Points)
Prove that the integer programming formulation models the uniform labeling problem.

Consider now the following algorithm. First, the algorithm solves the linear programming relaxation of the integer program above. The algorithm then proceeds in phases. In each phase, it picks a label $i \in L$ uniformly at random, and a number $\alpha \in [0, 1]$ uniformly at random. For each vertex $v \in V$ that has not yet been assigned a label, we assign it label i if $\alpha \leq x_v^i$.

Exercise 7.2 (8 Points)
Suppose that vertex $v \in V$ has not yet been assigned a label. Prove that the probability that v is assigned label $i \in L$ in the next phase is exactly $x_v^i/|L|$, and the probability that it is assigned a label in the next phase is exactly $1/|L|$. Further prove that the probability that v is assigned label i by the algorithm is exactly x_v^i.

Exercise 7.3 (8 Points)
We say that an edge e is *separated by a phase* if both endpoints were not assigned labels prior to the phase, and exactly one of the endpoints is assigned a label in this phase. Prove that the probability that an edge e is separated by a phase is $\frac{1}{|L|}\sum_{i\in L} z_e^i$.

Exercise 7.4 (8 Points)
Prove that the probability that the endpoints of edge e receive different labels is at most $\sum_{i\in L} z_e^i$.

Exercise 7.5 (8 Points)
Prove that the algorithm is a 2-approximation algorithm for the uniform labeling problem.