Saarland Informatics Campus

Assignment 4, Approximation Algorithms
Summer term 2017

Tobias Mömke, Hang Zhou
http://www-cc.cs.uni-saarland.de/course/61/

Due: 23 May 2017

Exercise 4.1 (10 Points)
Show Lemma 2 of the lecture:
The cost of I is at most f times the minimum cost for the set cover problem.

Exercise 4.2 (10 Points)
Show Lemma 4 of the lecture:
The cost of I' is at most f times the minimum cost for the set cover problem.

Exercise 4.3 (20 Points)
Let $G = (A, B, E)$ be a bipartite graph; that is, each edge $(i, j) \in E$ has $i \in A$ and $j \in B$. Assume that $|A| \leq |B|$ and that we are given nonnegative costs $c_{ij} \geq 0$ for each edge $(i, j) \in E$. A *complete matching* of A is a subset of edges $M \subseteq E$ such that each vertex in A has exactly one edge of M incident on it, and each vertex in B has at most one edge of M incident on it. We wish to find a minimum-cost complete matching. We can formulate an integer program for this problem in which we have an integer variable $x_{ij} \in \{0, 1\}$ for each edge $(i, j) \in E$, where $x_{ij} = 1$ if (i, j) is in the matching and 0 otherwise. Then the integer program is as follows:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{(i,j)\in E} c_{ij} x_{ij} & \\
\text{subject to} \quad & \sum_{j\in B:(i,j)\in E} x_{ij} = 1, & \forall i \in A, \\
& \sum_{i\in A:(i,j)\in E} x_{ij} \leq 1, & \forall j \in B, \\
& x_{ij} \in \{0, 1\}, & \forall (i, j) \in E.
\end{aligned}
$$

Consider the linear programming relaxation of the integer program in which we replace the integer constraints $x_{ij} \in \{0, 1\}$ with $x_{ij} \geq 0$ for all $(i, j) \in E$.

a) Show that given any fractional solution to the linear programming relaxation, it is possible to find in polynomial time an integer solution that costs no more than the fractional solution. (Hint: Given a set of fractional variables, find a way to modify their values repeatedly such that the solution stays feasible, the overall cost does not increase, and at least one additional fractional variable becomes 0 or 1.) Conclude that there is a polynomial-time algorithm for finding a minimum-cost complete matching.

b) Show that if a solution x (a vector with components x_{ij} for every $(i, j) \in E$) is such that x cannot be expressed as $\lambda x' + (1 - \lambda)x''$ for $0 < \lambda < 1$ and feasible solutions x' and x'' distinct from x, then $x_{ij} \in \{0, 1\}$ for all $(i, j) \in E$.