1. Übungsblatt zu Holographic Algorithms

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First Assignment will be discussed June 14 during lecture time

Solving the assignments is voluntary.

Aufgabe 1.1 Prove the following statements:

a) Every plane graph on $n \geq 3$ vertices has at most $3n - 6$ edges.

b) A plane graph has $3n - 6$ edges iff every face is a triangle, that is, bounded by three edges.

c) The complete graph K_5 on five vertices is not planar.

d) The complete bipartite graph $K_{3,3}$ with 3 nodes on each side is not planar. (Hint: Recall that a graph is bipartite iff it has no odd cycles.)

Definition 1 *a) Let $G = (V, E)$ be a graph and let $\mathcal{V} = \{V_1, \dots, V_s\}$ be a partition of the vertices in V. Let*

$$\mathcal{E} = \{\{V_i, V_j\} \mid i \neq j \text{ and there are } x \in V_i \text{ and } v \in V_j \text{ such that } \{x, y\} \in E.$$

Any graph $(\mathcal{V}, \mathcal{E})$ obtained in this way is called a contraction of G.

b) A graph H is a minor *of a graph G if there is a subgraph S of G such that H is a contraction of S.*

c) A graph S is a subdivision *of a graph H if we can obtain S by replacing edges of H by independent paths between their end vertices.*

d) A graph H is a topological minor *of a graph G if there is a subdivision of H that is a subgraph of G*

Aufgabe 1.2 a) Prove that every topological minor is a minor.

b) Prove that every minor with maximum degree 3 is also a topological minor.

c) Let $G = (V, E)$ be a graph. The contraction of e is the process of forming a contraction of G where one element of the partition is the edge e and every other element is a single vertex. Prove the following: H is a minor of G iff we can obtain H from G by first deleting some edges and vertices and then performing some edge contractions.

Remark 1 *Kuratowski's theorem states that a graph G is planar iff it does neither contain K_5 nor $K_{3,3}$ as a minor.*

Aufgabe 1.3 (Kirchhoff's matrix tree theorem) Let $G = (V, E)$ be a graph. The degree matrix D of G is the diagonal matrix with the degrees of the vertices on the diagonal. The *Laplacian* of a graph G is defined by $L = D - A$, where A is the adjacency matrix of G. (Of course, when forming D and A we have to use the same ordering of the vertices.) The indidence matrix I is defined as follows: The rows are labeled by vertices and the columns by edges. For every edge $e = \{u, v\}$, where $u < v$ in our chosen ordering, the entry $I_{u,e} = 1$ and the entry $I_{v,e} = -1$. All other entries are zero.

a) Prove that $L = II^T$.

b) Let L' be a matrix obtained from L by deleting the first row and column of L. (In fact, you can delete any one row and any one column.) Prove that $\det L'$ is the number of spanning trees of G.
(Hint: Cauchy-Binet formula)

Aufgabe 1.4 A closed ordered walk (clow) in an edge weighted directed graph $G = (V, E)$ with n nodes is a sequence of nodes $(v_1, \dots, v_\ell)$ such that $(v_1, v_2), \dots, (v_{\ell-1}, v_\ell), (v_\ell, v_1) \in E$. We assume that the nodes are ordered. v_1 is always the smallest node of the clow and it may appear only once opposed to all other nodes. ℓ is the length of the clow. A clow sequence is a sequence $C = (c_1, \dots, c_k)$ of clows such that the heads of the clows are strictly increasing. The length of the a clow sequence is the sum of the lengths of its clows. The weight $w(c)$ of a clow c is the product of the weights of the edges appearing in it and the weight $w(C)$ of a clow sequence is the product of the weights of the clows. The sign of a clow sequence with k clows is $(-1)^{n+k}$.

a) Construct an involution I on the set of all clow sequences of length n such that for any clow sequence C of length n with at least one repeated vertex, C and $I(C)$ have the same weight but opposite signs.

b) Prove that for any matrix A (viewed as a weight function on a directed graph G),

$$\det A = \sum_{\text{clow sequences } C \text{ of length } n} \operatorname{sgn}(C) w(C).$$

c) Compute via dynamic programming the following quantities $[\ell, v, v_0, s]$. $[\ell, v, v_0, s]$ is the sum of the weights of all partial clow sequences of length ℓ ending in vertex v where the head of the current partial clow is v_0 and s is the sign of the finished clows so far. Here a partical clow sequence is a clow sequence in which the last clow might be unfinished, that is, not yet closed.

d) Give a polynomial time division-free algorithm for computing the determinant.